Beyond the VaR

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Value at risk (VaR) as a standard measure of market risks has been widely implemented by financial institutions. A natural question with respect to risk management relates to the profile of losses beyond the VaR. This question is especially relevant when the distribution of asset returns is fat-tailed, or when the position includes options.

This article uses the concept of BVaR in order to take into account the profile of losses beyond the VaR. Technically speaking, this corresponds to the statistical mean of the losses exceeding the VaR. While the VaR focuses on the frequency of extreme events, BVaR integrates both the frequency and the size of extreme events.

Value at risk (VaR) is now considered a standard measure of market risks, and has been widely implemented by financial institutions. The VaR of a market position is a single number attempting to summarize the risk of that position. It is defined as the worst expected loss of the position over a given period of time, at a given confidence level. For example, for a probability level of 99%, a VaR equal to $1 million means that the loss of the position should not exceed $1 million in 99 cases out of 100 on average.[1]

A natural question is how much a position can lose on the exceptional hundredth case. This simple question is the basis for the motivation of this article.

Assuming a Gaussian distribution for the returns on the position, VaR is a sufficient statistic to analyze the risk of a position, as it is directly related to the standard deviation of the distribution. For example, the VaR computed with a probability level of 99%, denoted by VaR(99%), is linked to the standard deviation σ by the relationship: VaR(99%) = 2.326σ.

In the Gaussian case, knowledge of the VaR also lets us know the profile of the losses beyond the VaR. That is, whatever the level of market volatility, the VaR computed with a probability level of 99.9%, which is equal to 3.090σ, can always be obtained from the VaR computed with a probability level of 99% by the relationship VaR(99.9%) = 1.328VaR(99%).

In general, however, VaR does not give a full picture of the risk of a position, as more information is needed to fully understand its risk characteristics. First, the distribution of basic asset returns themselves may not be Gaussian. In this case, for a given probability level, a non-Gaussian distribution may give the same VaR number as the Gaussian distribution, but may present a very different distribution of the losses exceeding the VaR. The losses exceeding the VaR may be concentrated near the VaR, as for the Gaussian distribution, or may be widely spread beyond the VaR.

Second, even if the distribution of basic asset returns is Gaussian, the distribution of the returns on the position itself may not be Gaussian if the position is non-linear because of the presence of options. For traders maximizing their return under a VaR constraint, Vorst [2000] shows that it is optimal to implement strategies betting on extreme events and generating great losses exceeding the VaR.

Even rational behavior may lead to a profile of losses beyond the VaR very different from the one associated with the basic assets.[2] Basak and Shapiro [1999] show that an economic agent maximizing wealth (or utility) under a VaR constraint would suffer larger losses than an economic agent that does not consider this risk management technique.

The first part of this article develops a simple statistical framework to define the expected loss beyond the VaR. Called BVaR, this risk measure takes into account the profile of the losses beyond the VaR. While the VaR focuses on the frequency of extreme events, BVaR integrates both the frequency and the size of extreme events.

The second part presents an empirical study to illustrate this concept. VaR and BVaR are computed for both linear and non-linear positions in the U.S. equity market. BVaR is useful in risk management and financial regulation.

I. BVAR: THE EXPECTED LOSS BEYOND THE VaR

The theoretical framework used to define the concept of BVaR is presented first. We analyze application of the concept in two cases: when the distribution of basic asset returns is fat-tailed, and when the position is non-linear with the presence of options.

Theoretical Framework for BVaR

We look at evaluation of the risk beyond the VaR using simple statistics. The return on a market position over a given period of time is denoted by R. The associated probability density function is denoted by f_R and the cumulative distribution function by F_R. The VaR of a position computed with the statistical distribution F_R and for a probability level p is denoted by $\text{VaR}(F_R, p)$ to emphasize the dependence on these two inputs.

Expressed as a positive number and as a percentage of the initial value of the position, the VaR of the position is given by

$$\text{VaR}(F_R, p) = -F_R^{-1}(1-p) \tag{1}$$

The risk beyond the VaR is measured by the average of the losses exceeding the VaR as suggested in Longin [1997a].[3] Denoted by BVaR, this measure corresponds to the expectation of the return R, conditional on its being lower than –VaR. Like the VaR, the BVaR of a position depends on the statistical distribution F_R and on the probability level p. It is given by

$$\begin{aligned} \text{BVaR}(F_R, p) &= -E(R \mid R < -\text{VaR}) \\ &= -\frac{\int_{-\infty}^{-\text{VaR}} x\, f_R(x)dx}{F_R(-\text{VaR})} \end{aligned} \tag{2}$$

BVaR incorporates both the frequency of the losses beyond the VaR (in the denominator) and the size of the losses beyond the VaR (in the numerator) by taking into account the first moment of the distribution of the losses exceeding the VaR.

One difficulty in implementing tail-based risk measures such as BVaR is their poor statistical properties. For example, the BVaR estimated with the historical distribution can involve only a few observations ([1 – p]% of the database with p being usually close to one), and may then present a high estimation risk.

From a financial point of view, the BVaR measure may be appealing for several reasons. As shown in Artzner et al. [1999], it is a "coherent" measure of risk. Specifically, unlike VaR, BVaR satisfies the subadditivity property; that is, the risk of a global position is less than the sum of the risks of different elements of the position.

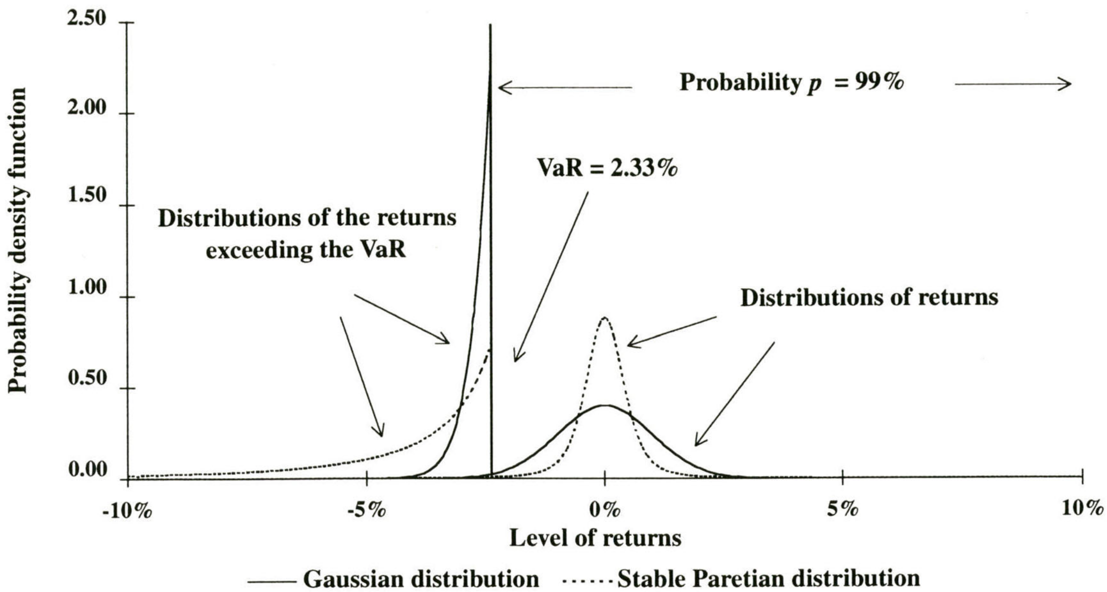
Basak and Shapiro [1999] also show that a BVaR sort of constraint for investors would lead to optimal strategies for which the magnitude of the extreme losses is under control. Their results based on simulations show that considering the first moment of the distribution of the losses beyond the VaR, i.e., the expected loss beyond the VaR or BVaR, may be sufficient to obtain a desirable loss profile. Further research in this direction would certainly be fruitful to understand how tail-based risk measures affect financial decisions.

BVaR and Fat-Tailed Distributions

A well-known fact is that most of the distributions of asset returns are fat-tailed, meaning that the number of observations contained in the tails (corresponding to large negative and positive returns) is actually higher than predicted by the Gaussian distribution. Such a fact can be rigorously assessed using extreme value theory (see Longin [1993] for a presentation). The degree of fatness of a distribution is characterized by a parameter called the *tail*

EXHIBIT 1

Distributions of Returns Exceeding VaR for Thin-Tailed and Fat-Tailed Distributions



index. According to the tail index value, negative, null, or positive, the distribution is fat-tailed, thin-tailed, or bounded. For example, the Gaussian distribution implies a tail index value equal to zero.

Jansen and De Vries [1991], Loretan and Phillips [1994], Longin [1996], and Booth et al. [1997] find a fat-tailed distribution for the U.S. equity market. Longin and Solnik [2001] have recently extended empirical studies focused on the U.S. to other major equity markets, and find that the shape of the distribution tails varies from one market to another. Similar results are obtained for foreign exchange rates by Koedijk, Schafgans, and De Vries [1990] and Longin [1997b], for interest rates by Boulier, Dalaud, and Longin [1997], for commodity markets by Longin [1999], and for emerging markets by Legras [2000].

When the distribution of the return on the position is fat-tailed, two statistical distributions of asset returns can lead to the same VaR number but to different profiles of the losses exceeding the VaR. Exhibit 1 represents the distributions of returns exceeding the VaR obtained from two distributions: the thin-tailed Gaussian distribution and the fat-tailed stable Paretian distribution.[4]

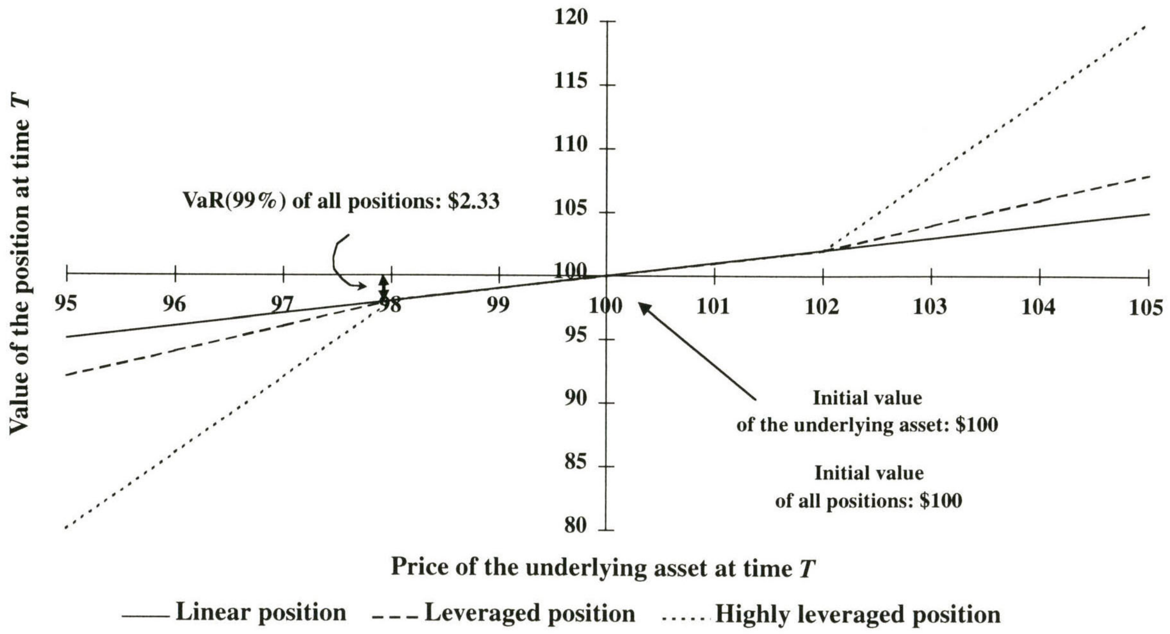
The two distributions are calibrated so that they give the same VaR number for a probability level of 99%: \$2.33 for a position with an initial value of \$100, or equivalently 2.33% of the initial value of the position. Note that the losses exceeding the VaR (lower than –VaR) are concentrated near the VaR for the Gaussian distribution, while they are widely spread beyond the VaR for the stable Paretian distribution. This result can be quantified by computing the expected loss beyond the VaR. The BVaR is higher for the fat-tailed stable Paretian distribution (3.22%) than for the thin-tailed Gaussian distribution (2.66%).

For a given distribution of returns, the BVaR can always be computed by simulation from Equation (2) as a function of the VaR. An asymptotic relationship (valid for high levels of the probability used to compute the VaR) can sometimes be derived analytically. For example, for a standard Gaussian distribution (with the mean equal to zero and the variance equal to one), it can be shown that the asymptotic relationship between BVaR and VaR is given by (see Embrechts, Klüppelberg, and Mikosch [1997, pp. 160-162]):

$$\text{BVaR} \approx \text{VaR} + \frac{1}{2}\frac{1}{\text{VaR}} \tag{3}$$

Equation (3) implies that, assuming the thin-tailed Gaussian distribution for the returns on the position, the losses beyond the VaR are concentrated near the VaR. Moreover, the higher the VaR, the more concentrated the losses beyond the VaR as the difference (BVaR – VaR) converges toward zero.

EXHIBIT 2
Value of Linear and Non-Linear Positions



For a standard stable Paretian distribution (with the location parameter equal to zero, the scale parameter equal to one, the skewness parameter equal to zero and a characteristic exponent α, with $\alpha > 1$), the relationship between BVaR and VaR is given by

$$\text{BVaR} \approx \text{VaR} + \frac{\text{VaR}}{\alpha - 1} \tag{4}$$

Equation (4) implies that, assuming the fat-tailed stable Paretian distribution for returns, the losses beyond the VaR are widely spread beyond the VaR. Moreover, the higher the VaR, the more spread the losses beyond the VaR as the difference (BVaR – VaR) diverges toward infinity. This result also depends on the degree of fatness of the distribution tail measured by the characteristic exponent. The fatter the tail (i.e., the lower the characteristic exponent), the more spread the losses beyond the VaR.

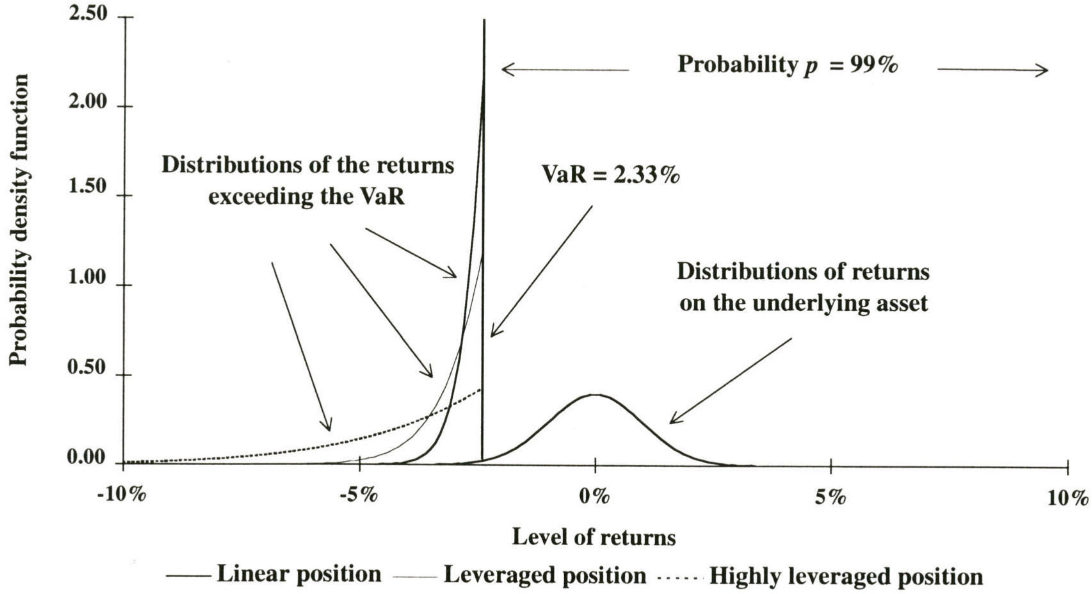
BVaR and Non-Linear Positions

For a non-linear position, or a position evolving over time according to a dynamic strategy that replicates a non-linear pay-off, BVaR is also a useful measure for analyzing the risk beyond the VaR. As in Vorst [2000], we consider here a position with options. Let us build a non-linear position with a long position in the underlying asset, selling out-of-the-money put options and buying out-of-the-money call options. More precisely, the position is short N^{put} put options and long N^{call} call options with strike prices denoted by K^{put} and K^{call}. The purchase of the call options is assumed to be financed by the sale of the put options ($N^{put}P = N^{call}C$). To focus on the effect of non-linearity on VaR and BVaR, the maturity of the call and put options, denoted by T, is assumed to be equal to the holding period used to compute the VaR.[5]

The strike price for the put options is chosen to be equal to or lower than (100 – VaR)% of the initial value of the underlying asset (where VaR is expressed as a positive number and as a percentage of the initial value of the position). This is modeled by $K^{put} \leq S_0(1 - \text{VaR}/100)$, where S_0 is the initial value of the underlying asset. In this case, the VaR of the non-linear position does not depend on the number of put options included in the position, as the probability level of losses due to the put options is equal to or lower than the probability level used to compute the VaR.

The value of the non-linear position is denoted by V. The change in value of the non-linear position, $V_T - V_0$, is equal to

EXHIBIT 3
Distributions of Returns Exceeding VaR for Linear and Non-Linear Positions



$$V_T - V_0 = S_T - N^{put}Max\left(K^{put} - S_T,\ 0\right) + N^{call}Max\left(S_T - K^{call},\ 0\right) - S_0 \qquad (5)$$

This example is graphed in Exhibit 2. Two types of position are considered: a linear position in the underlying asset, and non-linear positions that involve options. The linear position consists of a long position of \$100 invested in the underlying asset. Assuming a standard Gaussian distribution for the return on the underlying asset, the VaR of the linear position computed with a probability level of 99% is equal to \$2.33, or equivalently 2.33% of the initial value of the position.

The non-linear positions are composed of a long position in the underlying asset, writing out-of-the-money put options, and buying out-of-the-money call options. A leveraged position uses one put option, while a highly leveraged position uses five put options.

Exhibit 2 represents the value of the different positions. As the call options are assumed to be financed by selling the puts, the initial value of the non-linear positions is still equal to \$100. The value of the strike of the put options is chosen to be equal to \$97.67, that is to say, 2.33% lower than the initial value of the position. With such a value for the strike of the put options, the probability level of losses due to put options is exactly equal to the probability level used to compute the VaR (99%). In this way, the VaR of the non-linear positions is exactly equal to the VaR of the linear position (2.33).

Exhibit 3 represents the distribution of returns exceeding the VaR (lower than –VaR) for the three positions. As revealed in the graph, the losses exceeding the VaR are concentrated next to the VaR for the linear position (under the assumption of normality for the returns on the underlying asset), while the losses are widely spread beyond the VaR for the non-linear positions.[6]

Moreover, the dispersion of the losses beyond the VaR increases with the degree of leverage. This result can be quantified by computing the expected loss beyond the VaR. The BVaR is higher for the non-linear positions (3.00% of the initial value for a leveraged position and 4.36% for a highly leveraged position) than for the linear position (2.66%). Although the VaR is identical for the three positions, there is a difference in BVaR between linear and non-linear positions, and this difference increases with the degree of leverage of non-linear positions.

II. APPLICATION TO THE U.S. EQUITY MARKET

We demonstrate this theoretical framework by application to various positions in the U.S. equity market.

EXHIBIT 4

VaR and BVaR for Linear Positions in S&P 500 Index Computed with One-Day Returns

Panel A. Long Position

	p = 0.9			p = 0.99			p = 0.999		
	VaR	BVaR	(BVaR- VaR)/VaR	VaR	BVaR	(BVaR - VaR)/VaR	VaR	BVaR	(BVaR - VaR)/VaR
Historical	0.89	1.47	65.17	2.16	3.08	42.59	3.91	6.97	78.26
Extreme-Value	0.69	2.52	265.22	1.55	3.00	93.55	3.81	6.63	74.02
Unconditional Gaussian	1.04	1.44	38.46	1.92	2.21	15.10	2.57	2.81	9.34
EWMA	0.80	1.12	40.00	1.50	1.73	15.33	2.01	2.19	8.96
GARCH	0.81	1.13	39.51	1.52	1.76	15.79	2.05	2.24	9.27

Panel B. Short Position

	p = 0.9			p = 0.99			p = 0.999		
	VaR	BVaR	(BVaR - VaR)/VaR	VaR	BVaR	(BVaR - VaR)/VaR	VaR	BVaR	(BVaR - VaR)/VaR
Historical	0.95	1.53	61.05	2.25	2.99	32.89	4.09	4.98	21.76
Extreme-Value	1.15	2.42	110.43	1.68	2.85	69.64	3.70	4.89	32.16
Unconditional Gaussian	1.14	1.54	35.09	2.03	2.31	13.79	2.67	2.91	8.99
EWMA	0.90	1.22	35.56	1.60	1.83	14.38	2.11	2.30	9.00
GARCH	0.95	1.27	33.68	1.67	1.90	13.77	2.19	2.38	8.68

Data

The database consists of daily closing prices of the Standard & Poor's 500 index over the period January 1962–December 1999 consisting of 9,495 observations. These data are widely available and used here to allow an easy replication of the results.

Returns are computed for two frequencies: daily and biweekly. Given the daily frequency of the database, the use of one-day returns leads to the maximum number of observations that can be used to estimate both VaR and BVaR. Ten-day returns are also used here, as the regulation on market risks imposes the choice of this return frequency for internal models developed by financial institutions.[7]

Statistical Models

Statistical models used to compute the VaR and BVaR of a position can be distinguished in many ways. As explained in Longin [2000], statistical models can be classified according to the part of the distribution that is modeled. The classic approach consists of modeling the whole distribution of all returns, while the extreme-value approach focuses on the distribution of extreme returns. Statistical models can also be differentiated according to their time-varying properties. Unconditional distributions produce the same VaR and BVaR numbers, whatever the market conditions at the reporting date, while the VaR and BVaR given by conditional processes depend on the current market conditions (especially on the level of market volatility).

Several statistical models are used in the empirical study. They include three unconditional distributions: the historical distribution of returns, the asymptotic distribution of extreme returns, and the Gaussian distribution of returns; and two conditional Gaussian processes: the exponential weighted moving average (EWMA) for the variance process used in RiskMetrics™ [1995], and the generalized autoregressive conditional heteroscedastic (GARCH) process for the variance introduced by Engle [1982] and Bollerslev [1986]. Note that results obtained from conditional processes depend on the market conditions on the estimation date (such as the history of returns observed in the recent past). A detailed presentation of each statistical model along with the results of their estimation appears in the appendix.

The statistical models are used to compute both VaR and BVaR. In order to make meaningful compar-

EXHIBIT 5

VaR and BVaR for Linear Positions in S&P 500 Index Computed with Ten-Day Returns

Panel A. Long Position

	p = 0.9			p = 0.99			p = 0.999		
	VaR	BVaR	(BVaR - VaR)/VaR	VaR	BVaR	(BVaR - VaR)/VaR	VaR	BVaR	(BVaR - VaR)/VaR
Historical	3.02	5.04	66.89	7.00	10.15	45.00	n.c.	n.c.	n.c.
Extreme-Value	2.39	5.19	117.15	7.33	10.21	39.29	13.86	17.62	27.13
Unconditional Gaussian	3.34	4.76	42.51	6.48	7.49	15.59	8.77	9.60	9.46
EWMA	4.12	5.83	41.50	7.89	9.12	15.59	10.65	11.65	9.39
GARCH	4.10	5.80	41.46	7.86	9.08	15.52	10.60	11.60	9.43

Panel B. Short Position

	p = 0.9			p = 0.99			p = 0.999		
	VaR	BVaR	(BVaR - VaR)/VaR	VaR	BVaR	(BVaR - VaR)/VaR	VaR	BVaR	(BVaR - VaR)/VaR
Historical	3.82	5.53	44.76	7.96	9.55	19.97	n.c.	n.c.	n.c.
Extreme-Value	3.43	5.60	63.27	7.25	9.63	32.83	12.64	15.95	26.19
Unconditional Gaussian	4.35	5.77	32.64	7.48	8.50	13.64	9.77	10.60	8.50
EWMA	5.12	6.83	33.40	8.89	10.11	13.72	11.65	12.65	8.58
GARCH	5.12	6.83	33.40	8.88	10.10	13.74	11.63	12.62	8.51

N.c. Not calculable; too few observations.

isons between the different statistical models in terms of BVaR, differences between the VaR obtained from the different models have to be taken into account. To standardize the results, the ratio (BVaR – VaR)/VaR is computed. A low value (close to zero) for this ratio means that the losses beyond the VaR are concentrated near the VaR. A high value for this ratio means that the losses beyond the VaR are widely spread beyond the VaR.

Empirical Results for Linear Positions

Empirical results for linear positions in the U.S. equity market are given for two frequencies (one-day returns in Exhibit 4 and ten-day returns in Exhibit 5) and for both types of market position (a long position in Panel A and a short position in Panel B). As shown in Longin [2000], the VaR estimates vary widely from one model to another. The differences among models also tend to widen with the probability level, i.e., when we look farther into the tails.

Let us consider, for example, the results obtained for a long position in the S&P 500 index using one-day returns (Panel A of Exhibit 4). For a probability level of 99%, the VaR estimate for a position with an initial value of $100 is equal to $2.16 based on the historical distribution and to $1.92 based on the unconditional Gaussian distribution. For a probability level of 99.9%, these two estimates are, respectively, equal to $3.91 and $2.57 (a much greater difference).

Such a difference is well explained by the way the tail of a distribution is modeled. The distribution actually observed in financial markets presents fat tails, while the Gaussian distribution has thin tails (as reported in the appendix, the tail index estimate for the S&P 500 index is negative, implying a fat-tailed distribution for returns).

Turning to BVaR, there are also differences among the models, and these differences are even much greater than for the VaR. For a probability level of 99%, the BVaR estimate for a position with an initial value of $100 is equal to $3.08 based on the historical distribution and to $2.21 based on the unconditional Gaussian distribution (Panel A of Exhibit 4). For a probability level of 99.9%, the two estimates are, respectively, equal to $6.97 and $2.81 (a much greater difference).

Based on the historical distribution, the difference BVaR – VaR increases in absolute terms from $0.92 (3.08 – 2.16) at the 99% level to $3.06 (6.97 – 3.91) at

EXHIBIT 6

VaR and BVaR for Non-Linear Positions in S&P 500 Index with Put and Call Options

Panel A. Leveraged Position

	p = 0.9			p = 0.99			p = 0.999		
	VaR	BVaR	(BVaR - VaR)/VaR	VaR	BVaR	(BVaR - VaR)/VaR	VaR	BVaR	(BVaR - VaR)/VaR
Historical	3.02	5.78	91.39	8.90	15.20	70.79	n.c.	n.c.	n.c.
Extreme-Value	2.39	5.18	116.74	9.55	15.31	60.31	22.61	30.83	36.36
Unconditional Gaussian	3.34	5.12	53.29	7.85	9.88	25.86	12.43	14.09	13.35
EWMA	4.40	7.41	68.41	11.67	14.26	22.19	17.50	19.62	12.11
GARCH	4.51	7.71	70.95	12.11	14.77	21.97	18.10	20.27	11.99

Panel B. Highly Leveraged Position

	p = 0.9			p = 0.99			p = 0.999		
	VaR	BVaR	(BVaR - VaR)/VaR	VaR	BVaR	(BVaR - VaR)/VaR	VaR	BVaR	(BVaR - VaR)/VaR
Historical	3.02	8.75	189.74	16.50	35.40	114.55	n.c.	n.c.	n.c.
Extreme-Value	2.39	6.02	151.88	18.44	35.73	93.76	57.62	80.17	39.14
Unconditional Gaussian	3.34	6.56	96.41	13.33	19.43	45.76	27.07	32.06	18.43
EWMA	4.40	12.23	177.95	24.79	32.56	31.34	42.29	48.64	15.02
GARCH	4.51	13.05	189.36	26.12	34.10	30.55	44.09	50.61	14.79

N.c. Not calculable; too few observations.

the 99.9% level. Based on the unconditional Gaussian distribution, the difference BVaR – VaR decreases in absolute terms from \$0.29 (2.21 – 1.92) at the 99% level to \$0.24 (2.81 – 2.57) at the 99.9% level.

As the probability level increases, the BVaR computed with the historical distribution tends to diverge from the VaR, while the BVaR computed with the Gaussian distribution tends to converge toward the VaR as predicted by the theory.

Such an empirical property can also be appreciated in relative terms by considering the ratio (BVaR – VaR)/VaR. For a probability level of 99%, the value of this ratio is equal to 42.59% for the historical distribution and to 15.10% for the unconditional Gaussian distribution. Similarly, for a probability level of 99.9%, the values for this ratio are, respectively, 78.26% and 9.34%.

Considering all results presented in Exhibits 4 and 5, two groups of models emerge: on the one hand, the historical distribution and the extreme value distribution with high ratio values, and on the other hand, all the models based on normality with low ratio values. BVaR estimates based on the historical distribution and the extreme-value distribution are always relatively much higher than those based on statistical models assuming conditional or unconditional normality. In the case of the historical and extreme-value distributions, the ratio (BVaR – VaR)/VaR sometimes increases and sometimes decreases, while in the case of the Gaussian distribution the ratio always decreases rapidly toward zero.

In other words, a method assuming normality would underestimate not only the VaR but also the risk beyond the VaR.

Empirical Results for Non-Linear Positions

Empirical results for non-linear positions in the U.S. equity markets are given in Exhibit 6. Each position corresponds to a long position in the S&P 500 index, the selling of out-of-the-money put options on the S&P 500 index, and the buying of out-of-the-money index call options. These options are traded on the Chicago Mercantile Exchange (see the CME website for a description of the contracts). The number of index call options bought is calculated so that the buying price of the call options is equal to the selling price of the put options.

Two positions are considered: a leveraged position with one put option (Panel A) and a highly leveraged position with five put options (Panel B). The maturity of the options, which is equal to ten days, is chosen to correspond to the period used to compute both the VaR and the BVaR of the position. The building date of the position is December 6, 1999 (ten trading days before the maturity of the options). Both VaR and BVaR are estimated at that date.

The results are obtained using data as follows:

S&P 500 index value on Dec. 6, 1999:	**1,422.64**
Initial value of long position in the index:	**1,422.64**
Strike price of index put option:	**1,350.00**
Market price of index put option on Dec. 6, 1999:	**$2.00**
Number of index put options:	
1 (leveraged position) or	
5 (highly leveraged position)	
Strike price of index call option:	**1,500.00**
Market price of index call option on Dec. 6, 1999:	**$1.00**
Number of index call options:	
2 (leveraged position) or	
10 (highly leveraged position)	
Maturity date of index put and call options:	**Dec. 23, 1999**
Time to expiration: 10 trading days	

The index put options are out of the money. Their strike price (1,350.00) is 5.11% below the current index value (1,422.64). The probability for these options to finish out of the money at maturity is equal to 95.63% under the historical distribution, 96.93% under the unconditional Gaussian distribution, and around 94% under the conditional Gaussian processes, taking into account the current market conditions at December 6, 1999.

Let us first consider the results obtained for a leveraged position (Panel A of Exhibit 6). For a probability level of 90%, the VaR estimate for a position with an initial value of $100 is equal to $3.02 based on the historical distribution. This number is identical to the number obtained for a linear position (see Panel A of Exhibit 5) because the chosen probability level of 90% used to compute the VaR is too low to take into account the losses from in-the-money put options at maturity as indicated above. In other words, the VaR is unable to capture the risk associated with the put options. The BVaR measure deals with this problem: The BVaR of the leveraged position is equal to $5.78, and higher than the $5.04 obtained for the BVaR of the linear position.

This result is even more striking for a highly leveraged position (Panel B of Exhibit 6). For a probability level of 90%, the VaR estimate based on the historical distribution is still equal to $3.02, while the BVaR is now equal to $8.75. The value of the ratio (BVaR – VaR)/VaR computed for the historical distribution is equal to 91.39% for a leveraged position and to 189.74% for a highly leveraged position. Similar results are obtained with the other statistical models.

III. CONCLUSION

The risk measure BVaR allows us to summarize in a single number the profile of the losses beyond the VaR. Technically speaking, it corresponds to the statistical mean of the losses exceeding the VaR. The empirical results obtained here remind us that computation of both the VaR and the BVaR is very sensitive to the statistical model used to describe the behavior of asset returns. In particular, the assumption of normality leads to an underestimate of the true risk associated with extreme events.

As is true of any model, VaR does not provide a perfect description of reality. Financial institutions should be aware of its limitations. A practical way to deal with the risk beyond the VaR is to impose operational limits (such as in terms of number of contracts, nominal amount, sensitivities, or stop loss orders) in addition to VaR limits. BVaR, along with VaR, may also eventually become an effective risk management tool.

While the goal of VaR as a risk management tool is to control market risk during normal market conditions (say, 99% of the time), the objective of BVaR would be to control market risk during extraordinary market conditions (1% of the time). Such a control may be particularly important for positions invested in financial assets characterized by fat-tailed distributions or for positions including options.

Finally, the concept of BVaR may be of some interest for financial regulators as well. Since establishment of changes in regulations on market risk in 1996, financial institutions have been allowed to compute the regulatory capital for their market risks using their own internal models. As I argue in Longin [2000], basing the computation on classic VaR models does not take extraordinary market conditions explicitly into account.

A first approach to this problem would be to consider a VaR computed with a more conservative probability level such as 99.9% (instead of 99%) to better reflect the frequency of extreme events affecting market positions. Ideally computed using extreme value theory, this VaR would be similar to a stress value.

A second approach could be to relate the level of economic or regulatory capital to the BVaR, as this measure takes into account the whole distribution of extreme events in a more integrated manner. While the first approach considers only the frequency of extreme events, the second approach integrates both the frequency and the size of extreme events.

APPENDIX
Statistical Models

The Historical Distribution

The historical distribution is built over the entire period January 1962–December 1999. The period includes 9,494 observations of one-day returns and 949 observations of ten-day returns. For a probability level p, the VaR of a position corresponds to the (1 – p)% quantile of the historical distribution. It is given by

$$\mathrm{VaR}\left(F_R^{his}, p\right) = -\mathrm{Inf}\left(R_t^*,\ t/T \geq 1-p\right) \quad \text{(A-1)}$$

where F_R^{his} represents the historical distribution and $(R_t^*)_{t=1,T}$ the series of observed returns arranged in ascending order.

The BVaR of a position corresponds to the average of observed returns exceeding the VaR (lower than –VaR). It is given by

$$\mathrm{BVaR}\left(F_R^{his}, p\right) = -\frac{\sum_{1 \leq t \leq T(1-p)} R_t^*}{T(1-p)} \quad \text{(A-2)}$$

The Extreme-Value Distribution

The extreme-value distribution allows one to model the behavior of extremes of a random process. Extremes are defined as the highest observation (the maximum) and the lowest (the minimum) over a given period of time. The extreme-value theorem gives the form of the asymptotic distribution of standardized extremes. Three possible types of extreme-value distribution can be distinguished: Gumbel, Fréchet, and Weibull.

The minimum observed over n trading intervals, denoted by Z_n, is defined by $Z_n = \mathrm{Min}(R_1, R_2, \ldots, R_n)$. Following Gnedenko [1943] and Gumbel [1958], the asymptotic distribution of the standardized minimum $(Z_n - \beta_n)/\alpha_n$, denoted by F_z, is given by

$$F_Z(z) = 1 - \exp\left(-(1+\tau z)^{\frac{1}{\tau}}\right) \quad \text{(A-3)}$$

for $z < -1/\tau$ if $\tau < 0$ and for $z > -1/\tau$ if $\tau > 0$. The parameters α_n and β_n correspond to scale and location parameters. The parameter τ, called the tail index, determines the type of distribution: $\tau < 0$ corresponds to a Fréchet distribution, $\tau > 0$ to a Weibull distribution, and the intermediate case ($\tau = 0$) corresponds to a Gumbel distribution. The Gumbel distribution can be regarded as a transitional limiting form between the Fréchet and the Weibull distributions as $(1 - \tau z)^{1/\tau}$ is interpreted as e^{-z}. For small values of τ, the Fréchet and Weibull distributions are very close to the Gumbel distribution. The tail of the distribution of returns is either declining exponentially (Gum-

EXHIBIT
Estimation of Parameters

Frequency of Returns	Type of Extreme	Scale Parameter α_n	Location Parameter β_n	Tail Index τ
One-Day	Minimum Return	0.637 (0.078)	-1.690 (0.085)	-0.428 (0.117)
	Maximum Return	0.776 (0.076)	1.857 (0.100)	-0.128 (0.084)
Ten-Day	Minimum Return	1.858 (0.178)	-2.816 (0.237)	-0.128 (0.072)
	Maximum Return	1.398 (0.144)	3.749 (0.185)	-0.156 (0.099)

bel) or by a power (Fréchet), or remains finite (Weibull).

For the empirical study, the extreme-value distribution is estimated from extreme returns, which are selected over non-overlapping six-month periods. Estimation of the three parameters (α_n, β_n, and τ) is presented in the Exhibit.

A goodness-of-fit test is carried out to check the adequacy of the estimated extreme-value distribution to the observed distribution of extreme returns (see Longin [1996] for a description of the test).

As in Longin [2000], for a probability level p, the VaR is computed by the equation

$$\mathrm{VaR}\left(F_{Z_n}, p\right) = -F_{Z_n}^{-1}\left[(1-p)^n\right] \qquad \text{(A-4)}$$

where F_{Z_n} represents the asymptotic distribution of extreme returns selected over a period including n returns.

The BVaR is given by

$$\mathrm{BVaR}\left(F_{Z_n}, p\right) = -\frac{\int_{-\infty}^{-\mathrm{VaR}} x\, f_{Z_n}(x)dx}{F_{Z_n}(-\mathrm{VaR})} \qquad \text{(A-5)}$$

The Unconditional Gaussian Distribution

The unconditional Gaussian distribution is characterized by two parameters only: the mean and the variance. Over the period January 1962–December 1999, the empirical mean and variance of one-day returns are, respectively, equal to 0.050 and 0.721. For ten-day returns, the statistics are equal to 0.502 and 8.994. These numbers are used in Exhibits 3–6 to obtain the VaR and the BVaR of a position. The computation of these quantities is based on Equations (1) and (2), with F_R representing in this case a Gaussian distribution with constant mean and variance.

Conditional Gaussian Processes

Two conditional Gaussian processes are used: the exponential weighted moving average (EWMA) process and the generalized autoregressive conditional heteroscedastic (GARCH) process. Both processes describe the time-varying behavior of the variance of returns. More precisely, it is the expected variance of returns R_t observed at time t, computed one period before at time t – 1 and denoted by σ_t^2, that is modeled.

The EWMA equation for the expected variance is given by

$$\sigma_t^2 = (1-\lambda)\varepsilon_{t-1}^2 + \lambda\sigma_{t-1}^2 \qquad \text{(A-6)}$$

where the parameter λ, called the decay factor, reflects the persistence in volatility of both the latest innovation in returns ε_{t-1} and the past variance σ_{t-1}^2. The initial variance σ_0^2 is computed from the first observations of returns. An optimal value for the parameter λ is computed so that the variance process best describes the observed variance (measured by the squared innovations in returns). For the time series of the S&P 500 index, the decay parameter estimate is equal to 0.92 for one-day returns and to 0.89 for ten-day returns. These values are close to the ones applied to all time series by RiskMetrics: 0.94 for one-day returns and 0.97 for monthly returns.

The GARCH(1, 1) equation for the expected variance is given by

$$\sigma_t^2 = \alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \alpha_2\sigma_{t-1}^2 \qquad \text{(A-7)}$$

where parameter α_1 reflects the persistence in volatility of the latest innovation in returns ε_{t-1}, parameter α_2 reflects the persistence in volatility and of the past variance σ_{t-1}^2, and parameter α_0 is related to the unconditional level of volatility σ by $\alpha_0 = \sigma/(1 - \alpha_1 - a_2)$. Using one-day returns, the estimates of the three parameters α_0, α_1, and a_2 are, respectively, equal to 4.45 10^{-3} (6.10 10^{-4}), 0.077 (0.002), and 0.918 (0.003) with standard errors in parentheses. Using ten-day returns, the estimates of the three parameters a_0, a_1, and a_2 are 0.422 (0.165), 0.154 (0.025), and 0.811 (0.038).

For both conditional Gaussian processes, the VaR and BVaR of a position are computed (respectively) with Equations (1) and (2), with F_R representing in this case a Gaussian distribution with time-varying mean and variance. The variance is estimated by Equations (A-6) or (A-7). Using the estimated parameters of the volatility processes given above, the conditional variance of one-day returns at December 31, 1999, is equal to 0.469 for the GARCH process and to 0.445 for the EWMA process. For ten-day returns, the two statistics are equal to 12.941 and 13.001.

These numbers are used in Exhibits 4–6 to compute the VaR and BVaR of a position at a particular date, December 31, 1999. Although the parameters are estimated over the whole period (January 1962–December 1999), the conditional volatility estimates mainly reflect the market conditions a few months before the estimation date (around ten weeks for the EWMA process using one-day returns).

ENDNOTES

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[1]A general exposition is in Jorion [1997] and Dowd [1998]. Recent advances are given in Gaussel et al. [2000].

[2]Note that in well risk-managed financial institutions, operational limits (such as number of contracts, nominal amount, sensitivities, stop loss) in addition to VaR limits would constrain such behavior.

[3]References on the subject include Embrechts, Küppelberg, and Mikosch [1997]; Figlewski [1998], Artzner et al. [1999], Basak and Shapiro [1999], and Frey and McNeil [2000]. Note that in insurance a similar measure usually called the "mean excess function" has long been used to model the expected claim size of insurance contracts.

[4]See Mandelbrot [1963] for a presentation of stable Paretian distributions.

[5]If call and put options with longer maturity were considered, other effects such as changes in the level of interest rates and implied volatility or the passage of time would modify the analysis.

[6]Note the similarity between Exhibits 1 and 3. A position invested in assets characterized by a fat-tailed distribution and a position including options both lead to distributions of the losses beyond the VaR different from that obtained for the basic case (a linear position assuming the thin-tailed Gaussian distribution). Yet the distribution of the losses beyond the VaR is fat-tailed in the first case while still thin-tailed (but more dispersed) in the second case.

[7]See "An Internal Model-Based Approach to Market Risk Capital Requirements" [1995], "Amendment to the Capital Accord" [1996], and "Credit Institutions" [1996].

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