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From value at risk to stress testing: The extreme value approach

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Abstract

This article presents an application of extreme value theory to compute the value at risk of a market position. In statistics, extremes of a random process refer to the lowest observation (the minimum) and to the highest observation (the maximum) over a given time-period. Extreme value theory gives some interesting results about the distribution of extreme returns. In particular, the limiting distribution of extreme returns observed over a long time-period is largely independent of the distribution of returns itself. In financial markets, extreme price movements correspond to market corrections during ordinary periods, and also to stock market crashes, bond market collapses or foreign exchange crises during extraordinary periods. An approach based on extreme values to compute the VaR thus covers market conditions ranging from the usual environment considered by the existing VaR methods to the financial crises which are the focus of stress testing. Univariate extreme value theory is used to compute the VaR of a fully aggregated position while multivariate extreme value theory is used to compute the VaR of a position decomposed on risk factors. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The contribution of this article is to develop a new approach to VaR: *the extreme value approach*. As explained in Longin (1995), the computation of capital requirement for financial institutions should be considered as an extreme value problem. The focus of this new approach is on the *extreme* events in financial markets. Extraordinary events such as the stock market crash of October 1987, the breakdown of the European Monetary System in September 1992, the turmoil in the bond market in February 1994 and the recent crisis in emerging markets are a central issue in finance and particularly in risk management and financial regulation. The performance of a financial institution over a year is often the result of a few exceptional trading days as most of the other days contribute only marginally to the bottom line. Regulators are also interested in market conditions during a crisis because they are concerned with the protection of the financial system against catastrophic events which can be a source of systemic risk. From a regulatory point of view, the capital put aside by a bank has to cover the largest losses such that it can stay in business even after a great market shock.

In statistics, extremes of a random process refer to the lowest observation (the minimum) and to the highest observation (the maximum) over a given time-period. In financial markets, extreme price movements correspond to market corrections during ordinary periods, and also to stock market crashes, bond market collapses or foreign exchange crises during extraordinary periods. Extreme price movements can thus be observed during usual periods corresponding to the normal functioning of financial markets and during highly volatile periods corresponding to financial crises. An approach based on extreme values then covers market conditions ranging from the usual environment considered by the existing VaR methods to the financial crises which are the focus of stress testing. Although the link between VaR and extremes has been established for a long time, none of the existing methods could deal properly with the modeling of distribution tails.

To implement the extreme value approach in practice, a parametric method based on "extreme value theory" is developed to compute the VaR of a position. It considers the distribution of extreme returns instead of the distribution of all returns. Extreme value theory gives some interesting results about the statistical distribution of extreme returns. In particular, the limiting distribution of extreme returns observed over a long time-period is largely independent of the distribution of returns itself.

Two cases are considered to compute the VaR of a market position: a fully aggregated market position and a market position decomposed on risk factors. The former case may be used for positions with few assets and a stable composition and the latter case for complex positions with many assets and a time-changing composition. The case of a fully aggregated market position is treated with the asymptotic *univariate* distribution of extreme returns while the case of a market position decomposed on risk factors involves the asymptotic *multivariate* distribution of extreme returns. Univariate extreme value theory deals with the issue of tail modeling while multivariate extreme value theory addresses the issue of correlation or risk-aggregation of assets from many different markets (such as fixed-income, currency, equity and commodity markets) during extreme market conditions.

The first part of this article recalls the basic results of extreme value theory. The second part presents the extreme value method for computing the VaR of a market position. The third part illustrates the method by estimating the VaR and associated capital requirement for various positions in equity markets.

2. Extreme value theory

This section briefly discusses the statistical behavior of univariate and multivariate extremes. Both exact and asymptotic results pertaining to the distribution of extremes are presented.

2.1. The univariate distribution of extreme returns [1]

Changes in the value of the position are measured by the logarithmic returns on a regular basis. The basic return observed on the time-interval $[t-1, t]$ of length f is denoted by R_t. Let us call F_R the cumulative distribution function of R. It can take values in the interval (l, u). For example, for a variable distributed as the normal, one gets: $l = -\infty$ and $u = +\infty$. Let $R_1, R_2, \ldots, R_n$ be the returns observed over n basic time-intervals [0, 1], [1, 2], [2, 3],…, $[T-2, T-1]$, $[T-1, T]$. For a given return frequency f, the two parameters T and n are linked by the relation $T = nf$. Extremes are defined as the minimum and the maximum of the n random variables $R_1, R_2, \ldots, R_n$. Let Z_n denote the minimum observed over n trading intervals: $Z_n = \text{Min}(R_1, R_2, \ldots, R_n)$. [2] Assuming that

[1] The results of the basic theorem for independent and identically distributed (i.i.d.) variables can be found in Gnedenko (1943). Galambos (1978) gives a rigorous account of the probability aspects of extreme value theory. Gumbel (1958) gives the details of statistical estimation procedures and many illustrative examples in science and engineering. Applications of extreme value theory both in insurance and finance can be found in Embrechts et al. (1997) and Reiss and Thomas (1997). Leadbetter et al. (1983) give advanced results for conditional processes.

[2] The remainder of the article presents theoretical results for the minimum only, since the results for the maximum can be directly deduced from those of the minimum by transforming the random variable R into $-R$, by which minimum becomes maximum and vice versa as shown by the following relation: $\text{Min}(R_1, R_2, \ldots, R_n) = -\text{Max}(-R_1, -R_2, \ldots, -R_n)$.

returns R_t are independent and drawn from the same distribution F_R, the exact distribution of the minimal return, denoted by F_{Z_n}, is given by

$$F_{Z_n}(z) = 1 - (1 - F_R(z))^n. \quad (1)$$

The probability of observing a minimal return above a given threshold is denoted by p^{ext}. This probability implicitly depends on the number of basic returns n from which the minimal return is selected (to emphasize the dependence of p^{ext} on the variable n, the notation $p^{\text{ext}}(n)$ is sometimes used in this article). The probability of observing a return above the same threshold over one trading period is denoted by p. From Eq. (1), the two probabilities, p^{ext} and p, are related by the equation: $p^{\text{ext}} = p^n$.

In practice, the distribution of returns is not precisely known and, therefore, if this distribution is not known, neither is the exact distribution of minimal returns. From Eq. (1), it can also be concluded that the limiting distribution of Z_n obtained by letting n tend to infinity is degenerate: it is null for z less than the lower bound l, and equal to one for z greater than l.

To find a limiting distribution of interest (that is to say a non-degenerate distribution), the minimum Z_n is reduced with a scale parameter α_n (assumed to be positive) and a location parameter β_n such that the distribution of the standardized minimum $(Z_n - \beta_n)/\alpha_n$ is non-degenerate. The so-called *extreme value theorem* specifies the form of the limiting distribution as the length of the time-period over which the minimum is selected (the variables n or T for a given frequency f) tends to infinity. The limiting distribution of the minimal return, denoted by F_Z, is given by

$$F_Z(z) = 1 - \exp\left(-(1 + \tau z)^{1/\tau} \right) \quad (2)$$

for $z < -1/\tau$ if $\tau < 0$ and for $z > -1/\tau$ if $\tau > 0$. The parameter τ, called the tail index, models the distribution tail. Feller (1971, p. 279) shows that the tail index value is independent of the frequency f (in other words, the tail is stable under time-aggregation). According to the tail index value, three types of extreme value distribution are distinguished: the Fréchet distribution ($\tau < 0$), the Gumbel distribution ($\tau = 0$) and the Weibull distribution ($\tau > 0$).

The Fréchet distribution is obtained for fat-tailed distributions of returns such as the Student and stable Paretian distributions. The fatness of the tail is directly related to the tail index τ. More precisely, the shape parameter k (equal to $-1/\tau$) represents the maximal order of finite moments. For example, if k is greater than one, then the mean of the distribution exists; if k is greater than two, then the variance is finite; if k is greater than three, then the skewness is well-defined, and so forth. The shape parameter is an intrinsic parameter of the distribution of returns and does not depend on the number of returns n from which the minimal return is selected. The shape parameter corresponds to the number of degrees of freedom of a Student distribution and to the characteristic exponent of a stable Paretian distribution.

The Gumbel distribution is reached for thin-tailed distributions such as the normal or log-normal distributions. The Gumbel distribution can be regarded as a transitional limiting form between the Fréchet and the Weibull distributions as $(1+\tau z)^{1/\tau}$ is interpreted as e^z. For small values of τ the Fréchet and Weibull distributions are very close to the Gumbel distribution.

Finally, the Weibull distribution is obtained when the distribution of returns has no tail (we cannot observe any observations beyond a given threshold defined by the end point of the distribution).

These theoretical results show the generality of the extreme value theorem: all the mentioned distributions of returns lead to the same form of distribution for the extreme return, the extreme value distributions obtained from different distributions of returns being differentiated only by the value of the scale and location parameters and tail index.

The extreme value theorem has been extended to conditional processes. For processes whose dependence structure is not "too strong", the same limiting extreme-value distribution F_Z given by Eq. (2) is obtained (see Leadbetter et al., 1983, ch. 3). Considering the joint distribution of variables of the process, the following mixing condition (3) gives a precise meaning to the degree of dependence

$$\lim_{l\to+\infty} |F_{i_1,i_2,\ldots,i_p,j_1,j_1,\ldots,j_q}(x_{i_1},x_{i_2},\ldots,x_{i_p},x_{j_1},x_{j_2},\ldots,x_{j_q}) - F_{i_1,i_2,\ldots,i_p}(x_{i_1},x_{i_2},\ldots,x_{i_p})F_{j_1,j_2,\ldots,j_q}(x_{j_1},x_{j_2},\ldots,x_{j_q})| = 0, \quad (3)$$

for any integers $i_1 < i_2 < \cdots < i_p$ and $j_1 < j_2 < \cdots < j_q$, for which $j_1 - i_p \geqslant l$. If condition (3) is satisfied, then the same limiting results apply as if the variables of the process were independent with the same marginal distribution (the same scale and location parameters α_n and β_n can be chosen and the same limiting extreme-value distribution F_Z is also obtained). With a stronger dependence structure, the behavior of extremes is affected by the local dependence in the process as clusters of extreme values appear. In this case it can still be shown that an extreme value modeling can be applied, the limiting extreme-value distribution being equal to

$$F_Z(z) = 1 - \exp\left(-(1+\tau z)^{\theta/\tau}\right),$$

where the parameter θ, called the extremal index, models the relationship between the dependence structure and the extremal behavior of the process (see Leadbetter and Nandagopalan, 1989). This parameter is related to the mean size of clusters of extremes (see Embrechts et al., 1997, ch. 8; McNeil, 1998). The extremal index θ verifies: $0 \leqslant \theta \leqslant 1$. The equality $\theta = 1$ is obtained in the cases of weak dependence and independence. In other cases, the stronger the dependence, the lower the extremal index.

Berman (1964) shows that the same form for the limiting extreme-value distribution is obtained for stationary normal sequences under weak assumptions on the correlation structure (denoting by ρ_m the correlation coefficient between R_t and R_{t+m}, the sum of squared correlation coefficients $\sum_{m=1}^{+\infty} \rho_m^2$ has to remain finite). Leadbetter et al. (1983) consider various processes based on the normal distribution: discrete mixtures of normal distributions and mixed diffusion jump processes all have thin tails so that they lead to a Gumbel distribution for the extremes. As explained in Longin (1997a), the volatility of the process of returns (modeled by the class of ARCH processes) is mainly influenced by the extremes. De Haan et al. (1989) show that if returns follow an ARCH process, then the minimum has a Fréchet distribution.

2.2. The multivariate distribution of extreme returns [3]

Let us consider a q-dimensional vector of random variables denoted by $R = (R^1, R^2, \ldots, R^q)$. The realization of the ith component observed at time t is denoted by R_t^i. Although the definition of extremes is natural and straightforward in the univariate case, many definitions can be taken in the multivariate case (see Barnett, 1976). In this study the multivariate minimum Z_n observed over a time-period containing n basic observations is defined as $\mathrm{Min}(R_1^1, R_2^1, \ldots, R_n^1)$, $\mathrm{Min}(R_1^2, R_2^2, \ldots, R_n^2), \ldots, \mathrm{Min}(R_1^q, R_2^q, \ldots, R_n^q)$. The multivariate minimal return corresponds to the vector of univariate minimal returns observed over the time-period.

As for the univariate case, for an i.i.d. process, the exact multivariate distribution of the minimum can be simply expressed as a function of the distribution of the basic variable. As in practice, we do not know the exact distribution, so we consider asymptotic results. We assume that there is a series of a vector of standardizing coefficients (α_n, β_n) such that the standardized minimum $(Z_n - \beta_n)/\alpha_n$ converges in distribution toward a non-degenerate distribution. The main theorem for the multivariate case characterizes the possible limiting distributions: a q-dimensional distribution F_Z is a limiting extreme-value distribution, if and only if, (1) its univariate margins F_Z^1, $F_Z^2, \ldots, F_Z^q$ are either Fréchet, Gumbel or Weibull distributions; and (2) there is a dependence function, denoted by d_{F_Z}, which satisfies the following condition:

$$F_Z(z^1, z^2, \ldots, z^q) = 1 - \left(F_Z^1(z^1) F_Z^2(z^2) \cdots F_Z^q(z^q)\right)^{d_{F_Z}(z^q - z^1, z^q - z^2, \ldots, z^q - z^{q-1})}. \quad (4)$$

[3] A presentation of multivariate extreme value theory can be found in Tiago de Oliveira (1973), Galambos (1978) and Resnick (1987). Tawn (1988) deals specifically with the bivariate case.

Unlike the univariate case, the asymptotic distribution in the multivariate case is not completely specified as the dependence function is not known but has to be modeled. Considering two extremes Z^i and Z^j, a simple model is the linear combination of the dependence functions of the two special cases of total dependence and asymptotic independence as proposed by Tiago de Oliveira (1973):

$$d_{F_{Z^i,Z^j}}(z^j - z^i) = \rho_{ij} \frac{\mathrm{Max}(1, e^{z^j - z^i})}{1 + e^{z^j - z^i}} + (1 - \rho_{ij}). \tag{5}$$

The coefficient ρ_{ij} represents the correlation between the extremes Z^i and Z^j.

In summary, extreme value theory shows that the statistical behavior of extremes observed over a long time-period can be modeled by the Fréchet, Gumbel or Weibull marginal distributions and a dependence function. This asymptotic result is consistent with many statistical models of returns used in finance (the normal distribution, the mixture of normal distributions, the Student distribution, the family of stable Paretian distributions, the class of ARCH processes...). The generality of this result is the basis for the extreme value method for computing the VaR of a market position, as presented in Section 3.

3. The extreme value method for computing the VaR of a market position

This section shows how extreme value theory can be used to compute the VaR of a market position. [4] The method for a fully aggregated position, as presented here, first involves the univariate asymptotic distribution of the minimal returns of the position. The method is then extended to the case of a position decomposed on risk factors. The VaR of the position is obtained with a risk-aggregation formula, which includes the following inputs: the sensitivity coefficients of the position on risk factors, the VaR of long or short positions in risk factors, and the correlation between risk factors during extreme market conditions. The issues of positions including derivatives and conditional VaR based on the extremes are also discussed.

3.1. The extreme value method for a fully aggregated position

The method is summarized in the flow chart in Fig. 1. Each step is detailed below:

[4] A general exposition of VaR is given in Wilson (1996), Duffie and Pan (1997) and Jorion (1997).

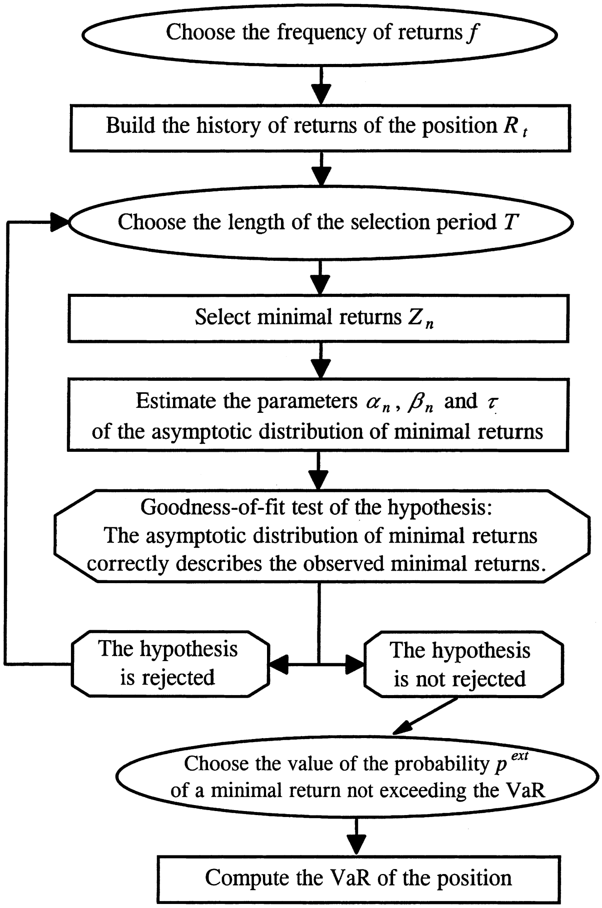

Fig. 1. Flow chart for the computation of VaR based on extreme values (this figure recalls the eight steps of the extreme value method for computing the VaR of a fully aggregated position).

Step 1: *Choose the frequency of returns f.* The choice of the frequency should be related to the degree of liquidity and risk of the position. For a liquid position, high frequency returns such as daily returns can be selected as the assets can be sold rapidly in good market conditions. The frequency should be quite high as extreme price changes in financial markets tend to occur during very short time-periods as shown by Kindleberger (1978). Moreover, low frequency returns may not be relevant for a liquid position as the risk profile could change rapidly. For an illiquid position, low frequency returns such as weekly or monthly returns could be a better choice since the time to liquidate the assets in good market conditions may be longer. However, the choice of a low frequency implies a limited number of (extreme) observations, which could impact adversely upon the analysis as extreme value theory is asymptotic by nature. The problem of infrequent trading characterizing assets of illiquid positions may be dealt with by some data adjustment as done in Lo and MacKinlay (1990) and Stoll and Whaley (1990). [5] The choice of frequency may also be guided or imposed by regulators. For example, the Basle Committee (1996a) recommends a holding period of 10 days.

Step 2: *Build the history of the time-series of returns on the position* R_t. For a fully aggregated position, a univariate time-series is used.

Step 3: *Choose the length of the period of selection of minimal returns T.* The estimation procedure of the asymptotic distribution of minimal returns does not only consider the minimal return observed over the entire time-period but several minimal returns observed over non-overlapping time-periods of length T. For a given frequency f, one has to determine the length of the period of selection of minimal returns, T or equivalently the number of basic returns, n, from which minimal returns are selected (as already indicated, the two parameters T and n are linked by the relation $T = nf$). The selection period has to satisfy a statistical constraint: it has to be long enough to meet the condition of application of extreme value theory. As this clearly gives an asymptotic result, extremes returns have to be selected over time-periods long enough that the exact distribution of minimal returns can be safely replaced by the asymptotic distribution.

Step 4: *Select minimal returns* Z_n. The period covered by the database is divided into non-overlapping sub-periods each containing n observations of returns of frequency f. For each sub-period, the minimal return is selected. From the first n observations of basic returns $R_1, R_2, \ldots, R_n$, one takes the lowest observation denoted by $Z_{n,1}$. From the next n observations $R_{n+1}, R_{n+2}, \ldots, R_{2n}$, another minimum called $Z_{n,2}$ is taken. From nN observations of returns, a time-series $(Z_{n,i})_{i=1,N}$ containing N observations of minimal returns is obtained. [6]

Step 5: *Estimate the parameters of the asymptotic distribution of minimal returns.* The three parameters α_n, β_n and τ of the asymptotic distribution of minimal returns denoted by $F_{Z_n}^{\text{asymp}}$ are estimated from the N observations of minimal returns previously selected. [7] The maximum likelihood method is used

[5] I am grateful to a referee for pointing out this issue.

[6] For a database containing N^{obs} observations of daily returns, for a frequency of basic returns f, and for a selection period of minimal returns containing n basic returns, the number of minimal returns N is equal to the integer part of $N^{\text{obs}}/f/n$.

[7] While classical VaR methods consider the information contained in the whole distribution, the method based on extreme values takes into account only the relevant information for the problem of VaR: the negative extremes contained in the left tail. The extreme value method focuses on the extreme down-side risk of the position instead of the global risk.

here as it provides asymptotically unbiased and minimum variance estimates. Note also that the maximum likelihood estimator can be used for the three types of extreme value distribution (Fréchet, Gumbel and Weibull) while other estimators such as the tail estimator developed by Hill (1975) are valid for the Fréchet case only. The extremal index θ may also be worth estimating if the data present strong dependence. Details of the estimation of the extremal index can be found in Embrechts et al. (1997, ch. 8).

Step 6: *Goodness-of-fit test of the asymptotic distribution of minimal returns.* This step deals with the statistical validation of the method: does the asymptotic distribution of minimal returns estimated in Step 5 describe well the statistical behavior of observed minimal returns? The test developed by Sherman (1957) and suggested by Gumbel (1958, p. 38), is based on the comparison of the estimated and observed distributions. The test uses the series of ordered minimal returns denoted by $(Z'_{n,i})_{i=1,N}$: $Z'_{n,1} \leqslant Z'_{n,2} \leqslant \cdots \leqslant Z'_{n,N}$. The statistic is computed as follows:

$$\Omega_N = \frac{1}{2}\sum_{i=0}^{N}\left|F_{Z_n}^{\text{asymp}}(Z'_{n,i+1}) - F_{Z_n}^{\text{asymp}}(Z'_{n,i}) - \frac{1}{N+1}\right|,$$

where $F_{Z_n}^{\text{asymp}}(Z'_{n,0}) = 0$ and $F_{Z_n}^{\text{asymp}}(Z'_{n,N+1}) = 1$.

The variable Ω_N is asymptotically distributed as a normal distribution with mean $(N/(N+1))^{N+1}$ and an approximated variance $(2e-5)/(e^2N)$, where e stands for the Neper number approximately equal to 2.718. The variable Ω_N can be interpreted as a metric distance over the set of distributions. A low value for Ω_N indicates that the estimated and observed distributions are near each other and that the behavior of extremes is well described by extreme value theory. Conversely, a high value for Ω_N indicates that the estimated and observed distributions are far from each other and that the theory does not fit the data. In practice, the value of Ω_N is compared with a threshold value corresponding to a confidence level (5% for example). If the value of Ω_N is higher than the threshold value, then the hypothesis of adequacy of the asymptotic distribution of minimal returns is rejected. The rejection can be explained by the fact that minimal returns have been selected over too short sub-periods. In other words, the number of basic returns from which minimal returns are selected is too small. Extreme value theory is, in fact, an asymptotic theory, and many basic observations should be used to select minimal returns such that the estimated distribution used is near the limit. If the hypothesis of adequacy is rejected, one has to go back to Step 3 and choose a longer selection period. If the value of Ω_N is lower than the given threshold, the hypothesis of adequacy is not rejected and one can go further to Step 7.

Step 7: *Choose the value of the probability p^{ext} of a minimal return not exceeding the VaR.* In the extreme value method, the usual definition for the probability is not used. [8] The reason is simple: we do not know of any model with a theoretical foundation that allows a link between the VaR and the probability of a return not exceeding the VaR (or more exactly – VaR, as VaR is usually defined to be a positive number). In the extreme value method, instead of using the probability related to a basic return, the probability related to a minimal return is used: for example, the probability of a minimal daily return observed over a semester being above a given threshold (the threshold value for a given value of the probability corresponding to the VaR number of the position). As explained in Section 2, for an independent or weakly dependent process, the two probabilities are related by: $p^{\text{ext}} = p^n$. [9] Note that when the distribution of returns is exactly known, the extreme value method is equivalent to any classical methods as they give the same VaR number for a given value of p or the associated value of p^{ext}. To emphasize the dependence of the VaR on the distribution and the probability used, the VaR obtained with a given distribution of returns F_R and a probability p is denoted by $\text{VaR}(F_R, p)$, and the VaR obtained with the associated exact distribution of minimal returns F_{Z_n} and a probability p^{ext} is denoted by $\text{VaR}(F_{Z_n}, p^n)$. Under the condition $p^{\text{ext}} = p^n$, we have $\text{VaR}(F_{Z_n}, p^n) = \text{VaR}(F_R, p)$.

The choice of the definition of the probability is guided by the statistical result concerning the extremes, presented is Section 2. The extreme value theorem shows that the link between the probability related to a minimal return and the VaR can be developed on theoretical grounds. The strength of the method is great as the asymptotic distribution of extremes is compatible with many statistical models used in finance to describe the behavior of returns.

The choice of the value of probability p^{ext} is arbitrary (as in other methods). However, several considerations can guide this choice: the degree of financial stability required by regulators (for example, the Basle Committee (1996a) presently imposes a value for probability p equal to 0.99 implying a value for probability p^{ext} equal to 0.99^n assuming weak dependence or independence of returns), the degree of risk accepted by the shareholders of financial institutions, and the communicability of the results in front of the Risk Committee of the banks. For example, the VaR computed with a value of 95% for the

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[8] The existing VaR methods of the classical approach use the probability of an unfavorable move in market prices under normal market conditions during a day or a given time-period, and then deduce the VaR with a statistical model. For example, the VaR computed by RiskMetrics™ developed by JP Morgan (1995) corresponds to the probability of observing an unfavorable daily move equal to 5% (equivalent to the probability p of a return not exceeding the VaR equal to 95%). In RiskMetrics™, the link between the probability and the VaR is realized with the normal distribution.

[9] The equation $p^{\text{ext}} = p^n$ is still valid in the case of weak dependence. In the case of strong dependence, one may use the following equation $p^{\text{ext}} = (p^n)^\theta$ involving the extremal index.

probability p^{ext} of a minimal return selected on a semester basis corresponds to the expected value of the decennial shock often considered by banks.

Step 8: *Compute the VaR of the position.* The last step consists of computing the VaR of the position with the asymptotic distribution of minimal returns previously estimated. The full model contains the following parameters: the frequency f and the number of basic returns n from which minimal returns are selected, the three parameters α_n, β_n and τ of the asymptotic distribution of minimal returns $F_{Z_n}^{asymp}$, and the probability p^{ext} of observing a minimal return not exceeding the VaR. For processes presenting strong dependence, if the value of probability p^{ext} is derived from the value of probability p using the equation $p^{ext} = (p^n)^{\theta}$, then the extremal index θ for minimal returns is also needed.

Considering the case of a fully aggregated position, the VaR expressed as a percentage of the value of the position is obtained from the estimated asymptotic distribution of minimal returns:

$$p^{ext} = 1 - F_{Z_n}^{asymp}(-\mathrm{VaR}) = \exp\left[-\left(1 + \tau\left(\frac{-\mathrm{VaR} - \beta_n}{\alpha_n}\right)\right)^{1/\tau}\right] \tag{6}$$

leading to

$$\mathrm{VaR} = -\beta_n + \frac{\alpha_n}{\tau}\left[1 - (-\ln(p^{ext}))^{\tau}\right]. \tag{7}$$

This "full" valuation method used to compute the VaR of a market position requires the construction of the history of returns of the entire position. For complex positions containing many assets or with a time-changing composition, it may be time-consuming to rebuild the history of returns of the position and re-estimate the asymptotic distribution of minimal returns every time the VaR of the position has to be computed. For this reason, it may be more efficient to decompose the position on a limited number of risk factors (such as interest rates, foreign currencies, stock indexes and commodity prices) and compute the VaR of the position in a simpler manner with a risk-aggregation formula. Such a method is presented next.

3.2. The extreme value method for a position decomposed on risk factors

The risk-aggregation formula relates the VaR of the position to the sensitivity coefficients of the position on risk factors, the VaR of long or short positions in risk factors and the correlation between risk factors. In this way, the computational work is reduced to the estimation of the multivariate distributions of both minimal and maximal returns of risk factors (which is done once for all) and to the calculation of the sensitivity coefficients of the position on risk factors (which is repeated every time the composition of the position changes).

A simple ad hoc risk-aggregation formula is used here to compute the VaR of a position. [10] Considering q risk factors, the VaR of a position characterized by the decomposition weights $(w_i)_{i=1,q}$, is given by

$$\mathrm{VaR} = \sqrt{\sum_{i=1}^{q}\sum_{j=1}^{q}\rho_{ij}\, w_i\, w_j\, \mathrm{VaR}_i\, \mathrm{VaR}_j}, \tag{8}$$

where VaR_i represents the VaR of a long or short position in risk factor i, w_i the sensitivity coefficient of the position on risk factor i and ρ_{ij} the correlation of extreme returns on long or short positions in risk factors i and j.

For each risk factor, the VaR of a long position, denoted by $\mathrm{VaR}_i^l(F_{Z_n}^{\mathrm{asymp}},\ p^{\mathrm{ext}})$, is computed with Eq. (7) using the parameters of the marginal distribution of minimal returns. Similarly, the VaR of a short position, denoted by $\mathrm{VaR}_i^s(F_{Y_n}^{\mathrm{asymp}},\ p^{\mathrm{ext}})$, is computed with the same equation using the parameters of the marginal distribution of maximal returns $F_{Y_n}^{\mathrm{asymp}}$.

Considering two risk factors i and j, four types of correlation of extreme returns ρ_{ij} can be distinguished according to the type of position (long or short) in the two risk factors. For a position long (short) in both risk factors i and j, the correlation ρ_{ij} corresponds to the correlation between the minimal (maximal) returns of risk factors i and j. For a position long (short) in factor i and short (long) in factor j, the correlation ρ_{ij} corresponds to the correlation between the minimal (maximal) return of risk factor i and the maximal (minimal) return in factor j. To emphasize the dependence on the type of position (long or short), the four correlation coefficients are denoted by ρ_{ij}^{ll}, ρ_{ij}^{ss}, ρ_{ij}^{ls} and ρ_{ij}^{sl}.

In the case of total dependence ($\rho_{ij} = 1$ for all i and j), the VaR of the position is equal to the sum of the weighted VaR of each risk factor, $\sum_{i=1}^{q} w_i\, \mathrm{VaR}_i$. In the case of independence ($\rho_{ij} = 0$ for all i and j, $i \neq j$), the VaR of the position is equal to the square root of the weighted sum of the squared VaR of each risk factor, $\sqrt{\sum_{i=1}^{q}(w_i\, \mathrm{VaR}_i)^2}$.

To summarize, for a position decomposed on a given set of q risk factors, the VaR can be formally written as

$$\mathrm{VaR}\left((w_i)_{i=1,q},\ \left(\mathrm{VaR}_i^l(F_{Z_n}^{\mathrm{asymp}},\ p^{\mathrm{ext}})\right)_{i=1,q},\ \left(\mathrm{VaR}_i^s(F_{Y_n}^{\mathrm{asymp}},\ p^{\mathrm{ext}})\right)_{i=1,q},\ \left(\rho_{ij}^{ab}\right)_{i,j=1,q}^{a,b=l \text{ or } s}\right).$$

[10] This formula is inspired by the formula used in the variance/covariance method. In the case of normality, the VaR obtained with the position decomposed on risk factors and based on the risk-aggregation formula (8) with the VaRs of risk factors and correlation coefficients classically computed corresponds to the VaR obtained with the fully aggregated position.

The method needs the estimation of $6q$ parameters of the marginal distributions of both minimal and maximal returns used to compute the $2q$ VaRs for long and short positions in risk factors, and the estimation of $4q$ correlation coefficients of the multivariate distribution of extreme returns used to aggregate the VaRs of risk factors.

The extreme value method takes explicitly into account the correlation between the risk factors during extreme market conditions. It has often been argued that there is a rupture of correlation structure in periods of market stress. For example, using multivariate GARCH processes, Longin and Solnik (1995) conclude that correlation in international equity returns tends to increase during volatile periods. Applying multivariate extreme value theory, Longin and Solnik (1997) find that correlation in international equity returns depends on the market trend and on the degree of market volatility. Correlation between extreme returns tends to increase with the size of returns in down-markets and to decrease with the size of returns in up-markets.

3.3. Positions with derivatives

The VaR of positions with classical options is usually computed by the delta-gamma method to take into account the non-linearity. The VaR of positions with more complex options is often obtained with a Monte Carlo simulation method since no analytical formula is available. With non-linearity, the tail behavior of the distribution is a critical issue. Although the extreme value method may be difficult to implement straightforwardly, extreme value theory may be very useful for determining a model for generating returns from the point of view of extreme events. For example, the non-rejection of the Fréchet distribution would suggest fat-tailed distributions such as a Student distribution or a GARCH process. The non-rejection of the Gumbel distribution for extreme returns would suggest thin-tailed distributions such as the normal distribution or a discrete mixture of normals. The non-rejection of the Weibull distribution for extreme returns would suggest that bounded distributions with no tails may be used to describe returns. [11] Parameters from these models may be estimated by considering the extremes. For example, the

[11] Note that if a given distribution of returns implies a particular (Fréchet, Gumbel or Weibull) distribution of extreme returns, the reverse is not true. For example, a Student distribution for returns implies a Fréchet distribution for extreme returns, but a Fréchet distribution for extreme returns does not necessarily imply a Student distribution for returns. Results about the extremes should be used to infer information about the tails of the distribution only. It should not be used to infer information about the whole distribution as the center of the distribution is not considered by extreme value theory.

number of degrees of freedom of a Student distribution and the degree of persistence in a GARCH process are directly related to the tail index value.

In an extreme value analysis using observed data, it is the historical distribution that is studied. However, for evaluating a position with derivatives, one may need a risk-neutral distribution. Using results for the normal distribution (Leadbetter et al., 1983, pp. 20–21), in a world à la Black–Scholes, the risk-neutral asymptotic distribution of extreme returns differs from the historical one by the value of the location parameter only: $\beta_n^* = \beta_n - (\mu r^0)$, where the asterisk refers to the risk-neutrality, and where μ and r^0 represent respectively the expected return and the risk-free interest rate over a time-period of length f. The scale parameter and the tail index are unaffected by the change of distribution: $\alpha_n^* = \alpha_n$ and $\tau^* = \tau = 0$. For other processes (such as a mixed diffusion process with jumps or a GARCH process), all three parameters may be affected by the change of probability.

3.4. Conditional VaR based on extreme values

Extreme value theory gives a general result about the distribution of extremes: the form of the limiting distribution of extreme returns is implied by many different models of returns used in finance (the normal distribution, a discrete mixture of normals, the Student distribution, the stable Paretian distribution, ARCH processes...). Such generality may also mean a lack of refinement. For example, as the asymptotic extreme value distribution is largely unconditional, the VaR given by the extreme value method is independent of the current market conditions. In practice, everyday risk management may be improved by taking into account the current market conditions. Thus it may be useful to investigate conditional VaR based on extreme values.

3.5. Related works

Boudoukh et al. (1995) also consider the distribution of extreme returns. They do not use the asymptotic results given by extreme value theory but derive exact results by assuming a normal distribution for returns. In a non-parametric setting Dimson and Marsh (1997) consider the worst realizations of the portfolio value to compute the risk of a position. Danielsson and De Vries (1997) and Embrechts et al. (1998) use a semi-parametric method to compute the VaR. In both works, the tail index is estimated with Hill's estimator, which can be used for the Fréchet type only. Note that the Weibull distribution is sometimes obtained, and the Gumbel distribution is often not rejected by the data (some foreign exchange rates and non-US equity returns for example).

4. Examples of application

The case of a fully aggregated position is illustrated with long and short positions in the US equity market. To illustrate the case of a position decomposed on risk factors, long, short and mixed positions in the US and French equity markets are considered.

Table 1
Estimation of the parameters of the asymptotic distributions of extreme daily returns on the S&P 500 Index observed over time-periods of increasing length: 1 week, 1 month, 1 quarter and 1 semester[a]

Length of the selection period	Scale parameter α_n	Location parameter β_n	Tail index τ	Goodness-of-fit test statistics
(A) *Minimal daily returns*				
1 week ($T = 5$)	0.492	−0.518	−0.183	2.770
($f = 1, n = 5, N = 1,585$)	(0.009)	(0.013)	(0.020)	[0.001]
1 month ($T=21$)	0.533	−1.074	−0.148	1.371
($f = 1, n = 21, N = 377$)	(0.023)	(0.030)	(0.031)	[0.085]
1 quarter ($T = 63$)	0.585	−1.451	−0.302	−1.040
($f = 1, n = 63, N = 125$)	(0.049)	(0.059)	(0.070)	[0.851]
1 semester ($T = 125$)	0.623	−1.726	−0.465	−0.229
($f = 1, n = 125, N = 63$)	(0.085)	(0.091)	(0.128)	[0.591]
(B) *Maximal daily returns*				
1 week ($T = 5$)	0.501	0.572	−0.084	3.291
($f = 1, n = 5, N = 1,585$)	(0.012)	(0.013)	(0.032)	[<0.001]
1 month ($T = 21$)	0.544	1.158	−0.140	0.938
($f = 1, n = 21, N = 377$)	(0.025)	(0.032)	(0.042)	[0.174]
1 quarter ($T = 63$)	0.705	1.597	−0.104	0.516
($f = 1, n = 63, N = 125$)	(0.053)	(0.071)	(0.066)	[0.303]
1 semester ($T = 125$)	0.845	1.985	−0.060	−0.139
($f = 1, n = 125, N = 63$)	(0.087)	(0.118)	(0.082)	[0.555]

[a] This table gives the estimates of the three parameters of the asymptotic distribution of extreme daily returns selected over time-periods of increasing length T as indicated in the first column. Estimation results are given for the distribution of minimal daily returns (Panel A) and for the distribution of maximal daily returns (Panel B). The scale and location parameters (α_n and β_n) and the tail index (τ) are estimated by the maximum likelihood method. Standard errors of parameters' estimates are given below in parentheses. The database consists of daily returns on the S&P 500 index over the period January 1962–December 1993 (7927 observations). Extreme daily ($f = 1$) returns are selected over non-overlapping sub-periods of length ranging from 1 week to 1 semester. A minimal (maximal) daily return corresponds to the lowest (highest) daily return on the S&P 500 index over a given sub-period. The number of selected extremes (N) is inversely related to the length of the selection period (T) or equivalently to the number of daily returns from which extreme returns are selected (n). The last column indicates the result of Sherman's goodness-of-fit test with the p-value (probability of exceeding the test-value) given below in brackets. The 5% confidence level at which the null hypothesis of adequacy (of the estimated asymptotic distribution of extreme returns to the empirical distribution of observed extreme returns) can be rejected, is equal to 1.645.

4.1. The case of a fully aggregated position

The extreme value method presented in Section 3.1 is now applied to the computation of the VaR of both long and short positions in the Standard and Poor's 500 index (this widely available data are used here to allow an easy replication of the results). Estimation results of the asymptotic distribution of extreme returns are first presented. VaRs are then computed for different values of the probability of an extreme return not exceeding the VaR. The sensitivity of VaR results to the frequency and to the length of the selection period, and the impact of the stock market crash of October 1987 on VaR results are also studied. The VaR given by the extreme value method is also compared with the VaR given by classical methods. Finally, capital requirements are computed in order to assess the regulation of market risks.

4.1.1. Estimation of the asymptotic distribution of extreme S&P 500 index returns

Results of the estimation of the parameters of the asymptotic extreme value distribution are given in Table 1 for minimal returns (Panel A) and for maximal returns (Panel B). Extreme daily ($f = 1$ day) returns are observed over time-periods ranging from one week ($T = 5$ days) to one semester ($T = 125$ days). The database consists of daily returns on the S&P 500 index over the period January 1962–December 1993. Looking at a long time-period (without important structural changes) allows consideration of a variety of market conditions that may occur again in the future. Returns are defined as logarithmic index price changes. Considering the results for minimal daily returns: the scale parameter increases from 0.492 to 0.623, indicating that the negative extremes are more and more dispersed; the location parameter increases (in absolute value) from 0.518 to 1.726, showing that the average size of negative extremes is larger and larger; the tail index value is always negative and is between −0.148 and −0.465, implying that the limiting distribution is a Fréchet distribution. [12] In other words, the asymptotic distribution of minimal daily returns shifts to the left and spreads while the shape of the distribution, and specially the way the left tail decreases, remains the same. As the extreme value distribution is a Fréchet distribution, Hill's estimator based on tail observations can be used for the tail index. Following the procedure used in Jansen and De Vries (1991), the tail index estimate for the left tail is equal to −0.290 with a standard error of 0.043.

[12] A Fréchet distribution for extreme returns in the US equity market has been found by Jansen and De Vries (1991), Loretan and Phillips (1994) and Longin (1996). Considering other international equity markets, Longin and Solnik (1997) find that all types of extreme value distribution (Fréchet, Gumbel and Weibull) are obtained.

A likelihood ratio test shows that the Gumbel distribution (and a fortiori the Weibull distribution) is always rejected in favor of the Fréchet distribution. The test is distributed as a chi-square variable with one degree of freedom (obtained by difference in the number of parameters in the Fréchet and Gumbel distributions). For example, in the case of minimal returns selected over a semester, the value of the test is equal to 40.545 with a p-value (probability to exceed the test value) less than 0.001.

Sherman's goodness-of-fit test indicates that the hypothesis of adequacy (of the estimated asymptotic distribution of extreme returns to the empirical distribution of observed returns) is not rejected at the 5% confidence level when extreme daily returns are selected over time-periods longer than a month. For extremes selected over a week, the test statistic is equal to 2.770 and is higher than the threshold value of 1.645 associated with a 5% confidence level. For extremes selected over a month, a quarter and a semester, the test statistics are respectively equal to 1.371, −1.040 and −0.229, and are lower than the threshold value. The exact distribution of minimal returns can then be safely replaced by the asymptotic distribution as long as minimal returns are selected over time-periods of length greater than a month. Similar comments apply to maximal daily returns. However, the right tail appears less heavy than the left tail, and the Gumbel distribution for maximal returns is not rejected by the data.

The impact of the stock market crash of October 1987 (the greatest observation associated with a record-low return of −22.90%) on the estimation of the parameters of the distribution of minimal returns is also investigated. Considering minimal daily returns selected over a semester, the whole sample contains 63 observations ($N = 63$). From Table 1A the parameters' estimates are with the standard error in parentheses: 0.623 (0.085) for the scale parameter, −1.726 (0.091) for the location parameter, and −0.465 (0.128) for the tail index. Removing the observation of the October 1987 crash from the sample (N is now equal to 62) and estimating the distribution of minimal returns again, the parameters' estimates are now: 0.604 (0.054) for the scale parameter, −1.748 (0.064) for the location parameter and −0.301 (0.093) for the tail index. The impact of the greatest observation is largest on the tail index, while the two standardizing coefficients are changed slightly. By dropping the observation of the stock market crash of October 1987, the estimated distribution appears to be less fat-tailed (the tail index is closer to zero). Although the difference is not statistically significant, the economic impact in terms of VaR and regulatory capital requirement may be worth studying.

The Basle Committee (1996a) allows banks to consider price shocks equivalent to a short holding period such as a day, but it recommends a holding period of 10 days. The behavior of the asymptotic distribution under temporal aggregation is not specified by extreme value theory although the tail index value should remain the same, as shown by Feller (1971, p. 279).

For this reason, asymptotic distributions of extreme returns of various frequencies are also estimated. Estimation results are given in Table 2 for minimal returns (Panel A) and for maximal returns (Panel B). Three values are used for frequency f: 1, 5 and 10 days. Empirically, the asymptotic distribution of minimal (maximal) returns spreads and shifts to the left (right). For example, considering minimal returns, the scale parameter is equal to 0.623 for one-day returns, 1.098 for five-day returns and 1.875 for 10-day returns, and the location parameter is equal to −1.726 for one-day returns, −2.746 for five-day returns and −3.244 for 10-day returns. Such a result was expected as low-frequency returns are more volatile than high-frequency returns. The tail index value is always negative (between −0.465 and −0.134)

Table 2
Estimation of the parameters of the asymptotic distributions of extreme returns on the S&P 500 Index of various frequencies: 1, 5 and 10 days[a]

Frequency of returns	Scale parameter α_n	Location parameter β_n	Tail index τ	Goodness-of-fit test statistics
(A) *Minimal returns*				
One-day returns ($f = 1$)	0.623	−1.726	−0.465	−0.229
($T = 125, n = 125, N = 63$)	(0.085)	(0.091)	(0.128)	[0.591]
Five-day returns ($f = 5$)	1.098	−2.746	−0.319	−0.599
($T = 125, n = 25, N = 63$)	(0.136)	(0.160)	(0.120)	[0.725]
Ten-day returns ($f = 10$)	1.875	−3.244	−0.134	0.499
($T = 120, n = 12, N = 63$)	(0.208)	(0.272)	(0.096)	[0.309]
(B) *Maximal returns*				
One-day returns ($f = 1$)	0.845	1.985	−0.060	−0.139
($T = 125, n = 125, N = 63$)	(0.087)	(0.118)	(0.082)	[0.555]
Five-day returns ($f = 5$)	1.207	3.033	−0.147	−1.929
($T = 125, n = 25, N = 63$)	(0.138)	(0.176)	(0.115)	[0.973]
Ten-day returns ($f = 10$)	1.606	3.834	−0.100	−0.243
($T = 120, n = 12, N = 63$)	(0.181)	(0.239)	(0.115)	[0.596]

[a] This table gives the estimates of the three parameters of the asymptotic distribution of extreme returns of various frequencies f as indicated in the first column. Estimation results are given for the distribution of minimal returns (Panel A) and for the distribution of maximal returns (Panel B). The scale and location parameters (α_n and β_n) and the tail index (τ) are estimated by the maximum likelihood method. Standard errors of parameters' estimates are given below in parentheses. The time-series of returns of various frequencies are built from the database of daily returns on the S&P 500 index over the period January 1962–December 1993. The parameter n is adjusted such that extreme returns of all three frequencies are selected over non-overlapping semesters ($T = 125$ for one-day and five-day returns and $T = 120$ for 10-day returns). The last column indicates the result of Sherman's goodness-of-fit test with the p-value (probability of exceeding the test-value) given below in brackets. The 5% confidence level at which the null hypothesis of adequacy (of the estimated asymptotic distribution of extreme returns to the empirical distribution of observed extreme returns) can be rejected, is equal to 1.645.

implying a Fréchet extreme value distribution for all frequencies. It seems to decrease slightly for minimal returns while it remains fairly stable for maximal returns.

The extremal index θ, which models the relationship between the dependence structure and the behavior of extremes of the process, is also estimated. Using the blocks method presented in Embrechts et al. (1997, pp. 419–421, Eq. (8.10)), the estimate of θ is equal to 0.72 when looking at minimal one-day returns selected over a semester, and equal to 0.73 when looking at maximal one-day returns (a threshold value of $\pm 5\%$ is used to define return exceedances). Higher values of θ are obtained when a lower frequency for returns is used: 0.84 for minimal 10-day returns selected over a semester and 0.92 for maximal 10-day returns. These estimates are close to the value $\theta = 1$ obtained for the case of weak dependence or independence.

4.1.2. VaR of long and short positions in the S&P 500 index

The estimations obtained above are now used to compute the VaR of positions in the S&P 500 index. Empirical results are reported in Table 3 for a long position (Panel A) and for a short position (Panel B). Two holding periods are considered: 1 day and 10 days ($f = 1$ and 10). Extremes returns are selected over time-periods of two different lengths: 1 quarter and 1 semester. The value of the probability p^{ext} of an extreme return not exceeding the VaR ranges from 50% to 99% (a higher probability value meaning a higher risk aversion or a higher degree of conservatism). It is important to compute the VaR for different probability values as it gives an idea of the profile of the expected loss beyond the VaR. [13] For example, considering a holding period of 1 day and minimal returns selected on a semester basis, the VaR is equal to \$1.98 for a long position of \$100 and for a probability value of 50%. In other words, there is one chance in two that the position loses more than \$1.98 in one trading session over a semester. The concept of mean waiting period (also called return period) is useful for interpreting the results. The mean waiting period is defined as the average time that one has to wait to see an observation exceeding a given threshold. The mean waiting period for a minimal return less than or equal to level z, denoted by $T(z)$ is equal to $1/F_{Z_n}^{\text{asymp}}(z)$ or $1/(1 - p^{\text{ext}})$, expressed in units of the selection period of minimal returns. The mean waiting period that one has to wait to observe a loss greater than \$1.98 is then equal to two semesters (or one year). For a value of 95% for the

[13] See Longin (1997b), Embrechts et al. (1998) and Artzner et al. (1999) for a measure of the expected loss beyond the VaR.

Table 3
VaR of long and short positions in the S&P 500 Index computed using the extreme value method[a]

Probability of not exceeding VaR	Holding period: 1 day		Holding period: 10 days	
	Selection period: 1 quarter	Selection period: 1 semester	Selection period: 1 quarter	Selection period: 1 semester
(A) *Long position*				
50%	2.18	1.98	4.12	3.72
(1 year)	[2.05, 2.30]	[1.88, 2.07]	[3.91, 4.33]	[3.50, 3.94]
75%	2.98	2.78	5.80	5.41
(2 years)	[2.74, 3.23]	[2.59, 2.97]	[5.49, 6.11]	[5.06, 5.77]
90%	4.21	4.20	7.78	7.72
(5 years)	[3.68, 4.75]	[3.72, 4.68]	[7.26, 8.29]	[7.05, 8.40]
95%	5.36	5.72	9.23	9.67
(10 years)	[4.42, 6.29]	[4.77, 6.66]	[8.48, 9.98]	[8.57, 10.77]
99%	9.07	11.76	12.63	15.19
(50 years)	[5.62, 12.52]	[7.27, 16.25]	[10.86, 14.41]	[11.85, 18.54]
(B) *Short position*				
50%	2.38	2.26	4.20	4.24
(1 year)	[2.26, 2.58]	[2.16, 2.37]	[4.03, 4.36]	[4.05, 4.44]
75%	3.10	3.04	5.54	5.70
(2 years)	[2.91, 3.30]	[2.89, 3.20]	[5.29, 5.79]	[5.41, 6.00]
90%	4.02	3.98	7.17	7.53
(5 years)	[3.66, 4.37]	[3.72, 4.24]	[6.74, 7.61]	[7.02, 8.04]
95%	4.73	4.69	8.41	8.95
(10 years)	[4.18, 5.28]	[4.31, 5.08]	[7.76, 9.06]	[8.17, 9.73]
99%	6.56	6.42	11.44	12.57
(50 years)	[5.10, 8.02]	[5.48, 7.37]	[9.82, 13.07]	[10.53, 14.61]

[a] This table gives the VaR of positions in the S&P 500 index computed using the extreme value method for various values of the probability p^{ext} of an extreme return not exceeding the VaR, as indicated in the first column. The corresponding waiting periods are given below in parentheses. The VaR of a long position (Panel A) is obtained with the estimated asymptotic distribution of minimal returns, while the VaR of a short position (Panel B) is obtained with the estimated asymptotic distribution of maximal returns. The VaR is computed for a position of \$100, or equivalently, the VaR is expressed as the percentage of the value of the position. Two holding periods are considered: 1 and 10 days ($f = 1$ and 10). Extremes returns are selected over non-overlapping time-periods of two different lengths: 1 quarter and 1 semester. The probability p^{ext} of observing an extreme return not exceeding the VaR depends on the length of the selection period (parameter T or n for a given frequency f): $p^{\text{ext}} = p^{\text{ext}}(n)$. The probability used to compute the VaR with extreme returns selected on a quarterly basis ($n = 63$) is related to the one used to compute the VaR computed with extreme returns selected on a semester basis ($n = 125$) by using the equation: $p^{\text{ext}}(63) = p^{\text{ext}}(125)^{63/125}$. The 50% confidence band is given below in brackets for each estimate of VaR. It is estimated from the quantiles of the estimated asymptotic distribution of extreme returns.

probability p^{ext} (equivalent to a mean waiting period of 20 semesters or 10 years), the VaR increases to \$5.72. A higher value for p^{ext} implies a higher VaR number. A confidence band measuring the uncertainty due to the estimation procedure of the asymptotic extreme value distribution can be computed. [14] For example, for a long position and a probability value of 95%, the 50% confidence band for the VaR estimate of \$5.72 is \$4.77, \$6.66. In other words there is a 50% chance for the VaR to be located between \$4.77 and \$6.66. The 90% confidence band is \$3.42, \$8.01.

Impact of the frequency: From the results reported in Table 3, the VaR numbers computed from returns with the lower frequency ($f = 10$ days) are always higher than the VaR computed from returns with the higher frequency ($f = 1$ day). They are around 90% higher in most of the cases. In order to compute the regulatory capital requirement, VaR numbers calculated according to shorter holding periods than 10 days have to be scaled up to 10 days, by the square root of the time factor. For example, VaR obtained from daily returns would have to be multiplied by a time factor of $\sqrt{10} \cdot 20$. For a long position and a probability value of 95%, the scaled VaR obtained with one-day returns is equal to \$18.09 ($= \sqrt{10} \cdot 5.72$) 21and is thus much higher than the VaR computed with 10-day returns (\$9.67). Such a difference suggests that the scaling factor proposed by the Basle Committee may be too high.

Impact of the length of the selection period: The VaR can be computed from the distribution of extreme returns selected over time-periods of different length as long as it fits well the empirical distribution of observed extremes. Of course, the value of the probability p^{ext} has to be adjusted as this parameter depends on the length of the selection period. For two selection periods containing n and n' basic returns, a simple adjustment rule may be $p^{\text{ext}}(n) = [p^{\text{ext}}(n')]^{n/n'}$. This rule is consistent with i.i.d. processes and also with processes presenting weak or strong dependence. VaR results previously presented for extreme returns selected over a semester are now compared with those obtained for extreme returns selected over a different time-period. The probability value of 95% used in the case of extreme returns selected over a semester corresponds to a probability value of 97.44% ($= 0.95^{63/125}$) for extreme returns selected over a quarter. Using the estimates of the parameters given in Table 1A the corresponding VaR numbers for a long position are \$5.36 for a selection period of a quarter. This number is not statistically different from the VaR obtained with a selection period of a semester (\$5.72). As similar VaR numbers are obtained, the extreme value method seems to be robust to the choice of the length of the selection period.

Impact of the dependence in the data: The relationship between the dependence in the data and the behavior of extremes is modeled with the extremal index θ. When the probability p^{ext} of an extreme return not exceeding the VaR

[14] The formula for the estimation error on the quantile estimation is given in Kendall (1994, pp. 358–359). The estimation risk for VaR is discussed in Jorion (1996). Techniques for verifying the accuracy of VaR can also be found in the recommendations of the Basle Committee on backtesting (1996b) and in Kupiec (1995).

is derived from the probability p of a basic return not exceeding the VaR, the equation $p^{ext} = (p^n)^\theta$ should be used in the case of strong dependence. Note that, as the extremal index is always lower than one, the VaR is always higher in the case of strong dependence than in the case of weak dependence or independence. Empirically, using the estimate of θ equal to 0.72 obtained with minimal daily returns selected over a semester, the equation $p^{ext} = p^n = 0.95$ becomes $p^{ext} = (p^n)^\theta = 0.95^{0.72} = 0.9637$ by taking into account the dependence. The corresponding VaR is equal to $6.60, compared to $5.72 obtained by assuming weak dependence or independence of returns. The impact of dependence seems to be less pronounced when a lower frequency for returns is employed. Using the estimate of θ equal to 0.84 obtained with minimal 10-day returns selected over a semester, the probability p^{ext} corrected for the effect of dependence is equal to 95.78% ($= 0.95^{0.84}$) and the corresponding VaR is equal to $10.58 compared to $9.67 obtained by assuming weak dependence or independence of returns. The difference in VaR results is not statistically significant but it may be judged quite large from an economic point of view (additional capital is always costly for financial institutions).

Impact of the stock market crash of October 1987: The impact of the stock market crash of October 1987 on VaR results is also investigated. Using the asymptotic distribution of minimal daily returns selected over a semester, the VaR of a long position is equal to $4.65 by excluding the crash from the set of minimal returns for the estimation, compared with $5.72 by including the crash. From a statistical point of view, the difference can be attributed to the tail index, whose value is larger when the return observation of the crash is included in the estimation.

4.1.3. Comparison with classical methods

The VaR given by the extreme value method, $\text{VaR}(F_{Z_n}^{asymp}, p^{ext})$ for a long position or $\text{VaR}(F_{Y_n}^{asymp}, p^{ext})$ for a short position, is now compared with the VaR given by classical methods, denoted by $\text{VaR}(F_R, p)$, where F_R is a particular distribution of returns. [15] To make VaR results given by both approaches directly comparable, it is assumed that the probabilities p^{ext} and p are related by the equation: $p^{ext} = p^n$, which is valid under the assumption of weak dependence and independence of returns. Four classical VaR methods are considered:

- $\text{VaR}(F_R^{his}, p)$ based on the historical distribution of returns, denoted by F_R^{his}.
- $\text{VaR}(F_R^{nor}, p)$ based on the normal distribution of returns, denoted by F_R^{nor}. [16]

[15] Former empirical studies on VaR include Beder (1995) and Jackson et al. (1997).

[16] Over the period January 1962–December 1993, the estimates of the mean μ and standard deviation σ of the normal distribution of returns are 0.027% and 0.883% for one-day returns, and 0.264% and 2.852% for 10-day returns.

- $\mathrm{VaR}(F_{R_t}^{\mathrm{GARCH}}, p)$ based on the conditional GARCH process. A GARCH(1, 1) is used here. Return R_t observed at time t is assumed to be drawn from a conditional normal distribution denoted by $F_{R_t}^{\mathrm{GARCH}}$. The conditional variance ${\sigma_t}^2$ of this distribution is given by $\sigma_t^2 = a_0 + a_1\,\varepsilon_{t-1}^2 + b_1\,\sigma_{t-1}^2$, where parameter a_1 represents the persistence in volatility of the latest squared innovation ${\varepsilon_t}_{-1}^2$, and parameter b_1 measures the persistence in volatility of the past variance ${\sigma_t}_{-1}^2$. [17]
- $\mathrm{VaR}(F_{R_t}^{\mathrm{EWMA}}, p)$ based on the exponentially weighted moving average (EWMA) process for the variance used in RiskMetrics™. Return R_t observed at time t is assumed to be drawn from a conditional normal distribution denoted by $F_{R_t}^{\mathrm{EWMA}}$. The conditional variance ${\sigma_t}^2$ of this distribution is given by $\sigma_t^2 = (1-\lambda)\varepsilon_{t-1}^2 + \lambda\,\sigma_{t-1}^2$, where the parameter λ, called the decay factor, reflects the persistence of volatility over time. [18] This process is an integrated GARCH(1, 1) process with the constraint $a_1 + b_1 = 1$.

The methods using the asymptotic extreme value distribution, the empirical distribution and the normal distribution are unconditional as they give the same results whatever the market conditions at the time of estimation. Conditional models such as the GARCH and EWMA processes account for the time-varying conditions of the market as they use a normal distribution with time-varying mean and variance. As a consequence, they lead to a VaR which reflects the degree of market volatility at the time of estimation.

Empirical VaR results for positions in the S&P 500 index are given in Table 4 for the various methods presented above. The VaR is computed for a long position (Panel A) and a short position (Panel B). Two holding periods are considered: 1 and 10 days ($f = 1$ and 10). Three values for the probability p^{ext} of an extreme return not exceeding the VaR are taken: 50%, 95% and 99% corresponding to waiting periods respectively equal to 1, 10 and 50 years.

Considering a long position and a holding period of 1 day, the VaR based on the historical distribution of returns observed over the period January 1962–December 1993 is equal to \$2.06 for a value of 50% for probability p^{ext} and to \$6.32 for a probability value of 95%. These numbers are close to those obtained with the extreme value distribution: \$1.98 and \$5.72. Such results are consistent with the good fit of the extreme value distribution to the data as previously discussed. Note that for very high probability values (99% for example), the VaR cannot be computed from the historical method because of

[17] The estimates of the three parameters a_0, a_1 and b_1 of the GARCH(1, 1) process are respectively equal to 0.004, 0.094 and 0.904 for one-day returns, and 0.488, 0.141 and 0.807 for 10-day returns.

[18] The estimate of the parameter λ of the EWMA process is equal to 0.93 for one-day returns and 0.86 for 10-day returns.

Table 4
VaR of long and short positions in the S&P 500 Index computed using the extreme value method and classical methods[a]

Methods used to compute VaR	Holding period: 1 day			Holding period: 10 days		
	Probability of not exceeding VaR			Probability of not exceeding VaR		
	50%	95%	99%	50%	95%	99%
(A) *Long position*						
$\mathrm{VaR}(F_{Z_n}^{\mathrm{asymp}}, p^{\mathrm{ext}})$	1.98	5.72	11.76	3.72	9.67	15.19
$\mathrm{VaR}(F_R^{\mathrm{his}}, p)$	2.06	6.32	n.c.	3.73	9.58	n.c.
$\mathrm{VaR}(F_R^{\mathrm{nor}}, p)$	2.22	2.93	3.31	4.27	7.24	8.70
$\mathrm{VaR}(F_{R_t}^{\mathrm{GARCH}}, p)$	1.01	1.34	1.52	2.12	3.73	4.53
$\mathrm{VaR}(F_{R_t}^{\mathrm{EWMA}}, p)$	0.93	1.24	1.40	1.95	3.41	4.12
(B) *Short position*						
$\mathrm{VaR}(F_{Y_n}^{\mathrm{asymp}}, P^{\mathrm{ext}})$	2.26	4.69	6.42	4.24	8.95	12.57
$\mathrm{VaR}(F_R^{\mathrm{his}}, p)$	2.41	4.65	n.c.	4.01	9.19	n.c.
$\mathrm{VaR}(F_R^{\mathrm{nor}}, p)$	2.27	2.98	3.36	4.80	7.77	9.23
$\mathrm{VaR}(F_{R_t}^{\mathrm{GARCH}}, p)$	1.09	1.43	1.60	2.81	4.43	5.22
$\mathrm{VaR}(F_{R_t}^{\mathrm{EWMA}}, p)$	0.99	1.29	1.45	2.48	3.93	4.25

[a] This table gives the VaR of positions in the S&P 500 index computed using various methods as indicated in the first column. The VaR is computed for a long position (Panel A) and for a short position (Panel B). Five statistical models are considered: the asymptotic distribution of extreme returns, the historical distribution of returns, the unconditional normal distribution of returns, and GARCH and EWMA processes of returns. The VaR is computed for a position of \$100, or equivalently, the VaR is expressed as the percentage of the value of the position. Two holding periods are considered: 1 and 10 days ($f = 1$ and 10). Extreme returns are selected over non-overlapping semesters ($n = 125$ and $T = 125$ for one-day returns and $n = 12$ and $T = 120$ for 10-day returns). Three values for the probability p^{ext} of an extreme return not exceeding the VaR are taken: 50%, 95% and 99% corresponding to waiting periods respectively equal to 1 year, 10 years and 50 years. For the classical VaR methods, the probability p of a return not exceeding the VaR is related to probability p^{ext} by the relation $p^{\mathrm{ext}} = p^n$, which is valid under the assumption of weak dependence or independence of returns. For the conditional models (GARCH and EWMA processes), the VaR is computed on December 31, 1993 (the last day of the database). For high values of probability p^{ext} (or p), the VaR is not calculable (n.c.) with the historical distribution because of the lack of data. All statistical models are estimated over the period January 1962–December 1993.

the lack of data. Such a drawback does not exist for the extreme value method which, as a parametric method, allows *out-of-sample* VaR computations.

The VaR based on the unconditional normal distribution of returns is equal to \$2.22 for a value of the probability p^{ext} equal to 50%. By comparison, the VaR based on the asymptotic extreme value distribution is equal to \$1.98. For low probability levels, the two methods then lead to similar results. As the normal distribution has thin tails, the VaR computed for more conservative probability levels is not much higher: for example, for a value of probability p^{ext} equal to 95%, it is equal to \$2.93, a number much lower than that given by the extreme value distribution (\$5.72). Such a result illustrates the problem of using the normal distribution in the presence of fat-tailed time-series: the VaR is largely underestimated by the normal distribution for conservative probability levels. As the estimation of the extreme value distribution lets the data speak for themselves, the *model risk* related to the choice of a particular distribution of returns is considerably reduced for the extreme value method.

The VaR based on the conditional GARCH and EWMA processes of returns is respectively equal to \$1.01 and \$0.93 for a value of the probability p^{ext} equal to 50% (VaR estimated on 31 December 1993). VaR numbers for higher probability levels are not much higher as the two processes assume that the conditional distribution of returns is normal. The two conditional models give low levels of VaR reflecting a low level of estimated volatility at the chosen time of VaR estimation. Higher levels of VaR would be obtained during more volatile periods, especially after a great market shock. Conditional methods are subject to the *event risk* due to unexpected changes in market conditions. By focusing on extreme events, the event risk is also considerably reduced for the extreme value method.

The extreme value approach is represented in Fig. 2a, in which the probability p^{ext} of a minimal return not exceeding the VaR, is graphically related to the VaR of the position through the estimated asymptotic distribution of minimal returns, $F_{Z_n}^{\text{asymp}}$. The classical approach (illustrated with the historical method) is represented in Fig. 2b, in which the probability p of a return not exceeding the VaR, is graphically related to the VaR through the historical distribution of all returns, F_R^{his}. Using the equation $p^{\text{ext}} = p^n$ to relate the two probabilities, the VaR given by the estimated asymptotic distribution of minimal returns, $\text{VaR}(F_{Z_n}^{\text{asymp}}, p^{\text{ext}})$, should be close to the VaR given by the historical distribution, $\text{VaR}(F_R^{\text{his}}, p)$. Any difference should be attributed to the estimation error.

4.1.4. Regulatory capital requirement and assessment of the regulation on market risks

In April 1995, the Basle Committee (1995) announced that commercial banks could use the results given by their internal risk-management model to compute the level of regulatory capital corresponding to their market risks. [19] The Basle Committee officially recognized VaR as sound risk-management practice as it adopted the formula for the level of capital C given by

$$C_t = \text{Max}\left(\text{VaR}_{t-1}, (M+m)\frac{1}{60}\sum_{j=1}^{60}\text{VaR}_{t-j}\right), \tag{9}$$

[19] A general discussion of capital requirement for financial institutions can be found in Merton and Perold (1993), Dewatripont and Tirole (1994) and Berger et al. (1995).

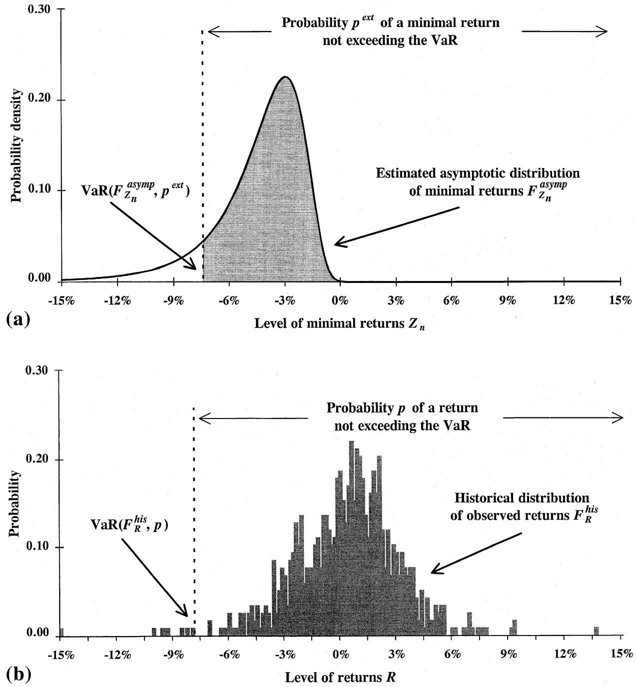

Fig. 2. a and b. The extreme value approach and the classical approach for the computation of VaR. These two figures illustrate graphically the computation of VaR based on two different approaches: the extreme value approach considering *extreme* returns only and the classical approach considering *all* returns. The estimated asymptotic distribution of minimal returns and the historical distribution of all returns are taken to implement each approach. Fig. 2a represents the estimated asymptotic distribution of minimal returns selected from n basic returns of frequency f. The gray area between the X axis, the curve of the probability density function and a vertical line breaking the X axis at the point of abscissa VaR (or more exactly – VaR, as VaR is usually defined to be a positive number) corresponds to the value of the probability $1 - p^{\text{ext}}$ of a minimal return not exceeding the VaR. Mathematically, the VaR obtained with the extreme value method, denoted by $\text{VaR}(F_{Z_n}^{\text{asymp}}, p^{\text{ext}})$, is equal to the inverse of the distribution of minimal returns, $F_{Z_n}^{\text{asymp}}$, evaluated at the point $1 - p^{\text{ext}}$. Fig. 2b represents the histogram of the historical distribution of observed returns of frequency f. The VaR obtained with the historical method, denoted by $\text{VaR}(F_R^{\text{his}}, p)$, is equal to the $(1 - p)$th quantile of the historical distribution F_R^{his}. Under the assumption of weak dependence or independence of returns, the probability p^{ext} is linked to the probability p by the relation: $p^{\text{ext}} = p^n$. The two figures use data from the S&P 500 index over the period January 1962–December 1993. The VaR is computed with a holding period of 10 days ($f = 10$) and a value for the probability of a 10-day return not exceeding the VaR, equal to 99% ($p = 0.99$) as defined by regulators (see Basle Committee, 1996a). For the extreme value method extreme 10-day returns are selected over a semester ($n = 12$ and $T = 120$). The value of the probability of an extreme return not exceeding the VaR is equal to 88.64% ($= 0.99^{12}$). The VaR numbers given by the extreme value method and the historical method are respectively equal to \$7.28 and \$7.97 for a long position of \$100 in the S&P 500 index.

where M is a multiplier component whose value is arbitrarily set at 3, m an additional component whose value, between 0 and 1, depends on the quality of the prediction of the internal model developed by the institution (see the recommendation on backtesting issued by the Basle Committee (1996b)), VaR_{t-j} the VaR computed by the internal model on day $t-j, t-1$ the estimation or reporting date and t the period of application. The VaR should be computed on a daily basis, using a 99th percentile, one-tailed confidence interval and price shocks equivalent to a holding period of 10 days.

The regulatory capital requirement computed with an internal model is equal to the maximum of the VaR computed at the reporting time and $(M+m)$ times the average of the VaR over the last 60 days preceding the reporting time. With the multiplier component M equal to 3 and the additional component m assumed to be equal to 0, the regulatory capital requirement is then equal to three times the VaR for internal models based on unconditional distributions. For a long position of \$100 in the S&P 500 index, the level of capital requirement given by the estimated asymptotic distribution of minimal returns is equal to \$21.84 (the value of the probability p^{ext} associated with the usual probability value p of 99% is equal to 88.64%). Similarly, the level of capital requirement given by the historical distribution of all returns is equal to \$23.91. By comparison, the lowest 10-day return on the S&P 500 index observed over the period January 1962–December 1993 is equal to −19.70% (during October 1987). The regulatory capital given by the extreme value and historical methods would have thus covered *all* losses observed during the entire period.

As done in Boulier et al. (1997), the results given by internal models can be compared with the rules of the standard method defined by the Basle Committee (1996a) and the European Commission (1996). According to the standard method, the minimum capital for equity position risk must be equal to 12%. This number corresponds to the sum of 8% for the "general market risk" of holding a long position in the market as a whole and 4% for the "specific risk" of holding a long position in each individual asset of a liquid portfolio. The results of this article show that regulatory capital levels obtained from the data themselves, either parametrically by the extreme value method or non-parametrically by the historical method, are far higher (almost double) than those obtained by the standard method proposed by the regulators. In terms of mean waiting period, a loss exceeding the capital given by the standard method (\$12) would be observed on average every 25 years, while a loss exceeding the capital implied by the extreme value method (\$21.84) based on Eq. (9) given by the Basle Committee would be observed on average every 300 years (the values of the mean waiting period have been obtained using the asymptotic extreme value distribution). This result suggests that for internal models taking explicitly into account extreme events, a value of 3 for the multiplier component M may be too high.

4.2. The case of a position decomposed on risk factors

The extreme value method presented in Section 3.2 is now applied to the computation of the VaR of various positions in US and French equity markets. Estimation results of the asymptotic bivariate distribution of extreme returns are first presented. VaR results are then discussed.

4.2.1. Estimation of the asymptotic bivariate distribution of extreme S&P 500 and SBF 240 index returns

The bivariate distribution of extreme returns is now estimated for the S&P 500 index and the SBF 240 index using 10-day returns over the period January 1976–December 1993. [20] The estimates of the scale and location parameters and the tail index (with standard error in parentheses) are respectively equal to 1.423 (0.253), −3.128 (0.283) and −0.393 (0.183) for the distribution of minimal 10-day returns on the S&P 500 index and 2.235 (0.321), −4.174 (0.421) and −0.142 (0.122) for the distribution of minimal 10-day returns on the SBF 240 index. Similarly for the distribution of maximal returns the estimates are 1.428 (0.229), 3.794 (0.281) and −0.240 (0.162) for the S&P 500 index and 1.357 (0.199), 4.560 (0.256) and −0.172 (0.126) for the SBF 240 index. For both equity markets the asymptotic distribution of extreme returns is a Fréchet distribution (although the Gumbel distribution may not be rejected in some cases).

The correlation between extreme returns in short or long positions in the S&P 500 and SBF 240 indexes is estimated from Eq. (5) using the regression method developed by Tiago de Oliveira (1974). The correlation between minimal returns of long positions in both indexes is equal to 0.418, the correlation between maximal returns of short positions in both indexes to 0.064, the correlation between minimal returns of a long position in the S&P 500 index and maximal returns of a short position in the SBF 240 to 0.185, and the correlation between maximal returns of a short position in the S&P 500 index and minimal returns of a long position in the SBF 240 to 0.095. The historical correlation computed with all returns is equal to 0.381. It is thus slightly lower than the correlation between minimal returns but much higher than the correlation between maximal returns.

4.2.2. VaR of long, short and mixed positions in S&P 500 and SBF 240 indexes

Three distributions are considered: the historical distribution (used as a benchmark), the extreme value distribution and the normal distribution.

[20] Low frequency returns are used here to avoid the problem of non-synchronous trading effects for stocks traded in different time-zones (see Hamao et al., 1990 for a discussion of correlation and spillover effects).

Long, short and mixed positions are constructed using the two basic investments (the S&P 500 index and the SBF 240 index) with different weights in these investments. The foreign exchange risk is assumed to be fully hedged. A position is described by the type of position in each index (long or short) and by the weights $(w, 1-w)$ in each index. When the parameter w is equal to 0% or 100%, a position is taken in one market only. In this case, any difference between the VaR computed with the extreme value or normal distributions and the VaR computed with the historical distribution, can be attributed to a "tail error" coming from a poor description of the distribution tails by the model. This error is likely due to estimation risk in the case of the extreme value distribution and model risk for the normal distribution. When the parameter w is between 0% and 100%, a position is taken in both markets and the correlation plays an important role. In this case, any difference between the VaR computed with the asymptotic extreme value distribution or the normal distribution and the VaR computed with the historical distribution can be attributed either to a tail error or to a correlation error coming from a poor description of the correlation between risk factors by the model (the risk-aggregation formula may also be misspecified).

Empirical results of VaR are reported in Table 5 for long positions (Panel A), for short positions (Panel B) and for mixed positions (Panels C and D). For example, for a long equally weighted ($w = 50\%$) position, the VaR is equal to \$7.62 with the historical distribution, \$7.39 with the extreme value distribution, and \$3.80 with the normal distribution. The VaR number obtained by the extreme value distribution has been computed as follows:

$$\sqrt{(0.50 \cdot 7.82)^2 + (0.50 \cdot 9.69)^2 + 2 \cdot 0.418 \cdot 0.50 \cdot 0.50 \cdot 7.82 \cdot 9.69},$$

where 0.50 represents the value of the weights of the portfolio in each index, 7.82 and 9.69 the VaR numbers of long positions in the S&P 500 index and in the SBF 240 index respectively, and 0.418 the correlation between minimal returns in the two indexes during extreme market conditions. The historical VaR is thus slightly underestimated by the extreme value method (a difference of −3.03%) and largely underestimated by the variance/covariance method (a difference of −50.10%). Similarly, for a short equally weighted position in both indexes, the VaR is equal to \$5.74 with the extreme value distribution, \$4.50 with the normal distribution and \$6.33 with the historical distribution. The historical VaR is underestimated by the extreme value method (a difference of −9.25%) and largely underestimated by the variance/covariance method (a difference of −28.84%). Note from Table 5 that the historical VaR is sometimes underestimated and sometimes overestimated by the extreme value method while it is always largely underestimated by the variance/covariance

Table 5
VaR computed for long, short and mixed positions in the S&P 500 and SBF 240 Indexes[a]

Position weights	Historical	Extreme value	Normal
(A) *Long positions in the S&P 500 and SBF 240 Indexes*			
(100, 0)	7.67	7.82 (+1.96)	3.13 (−59.19)
(75, 25)	7.14	7.22 (+1.14)	3.20 (−55.15)
(50, 50)	7.62	7.39 (−3.03)	3.80 (−50.10)
(25, 75)	9.06	8.28 (−8.64)	4.73 (−47.75)
(0, 100)	11.01	9.69 (−11.99)	5.84 (−46.96)
(B) *Short positions in the S&P 500 and SBF 240 Indexes*			
(100, 0)	8.74	7.73 (−11.56)	3.90 (−55.38)
(75, 25)	7.41	6.25 (−15.59)	3.90 (−47.40)
(50, 50)	6.33	5.74 (−9.25)	4.50 (−28.84)
(25, 75)	7.96	6.43 (−19.16)	5.52 (−30.62)
(0, 100)	8.97	8.02 (−10.59)	6.77 (−24.53)
(C) *Long position in the S&P 500 Index and short position in the SBF 240 Index*			
(75, 25)	5.40	6.35 (+17.73)	3.01 (−44.28)
(50, 50)	4.63	5.83 (+26.00)	3.84 (−16.88)
(25, 75)	6.71	6.48 (−3.41)	5.20 (−22.46)
(D) *Short position in the S&P 500 Index and long position in the SBF 240 Index*			
(75, 25)	6.65	6.36 (−4.34)	3.31 (−50.16)
(50, 50)	4.72	6.30 (+33.53)	3.56 (−24.42)
(25, 75)	5.65	7.58 (+34.25)	4.52 (−19.99)

[a] This table gives the VaR of market positions in the S&P 500 index (US equity market) and in the SBF 240 index (French equity market) with various position weights $(w, 1-w)$ as indicated in the first column. VaR numbers are computed for long positions (Panel A), short positions (Panel B) and mixed positions (Panels C and D). The VaR is computed for a position of \$100, or equivalently, the VaR is expressed as the percentage of the value of the position. Three statistical distributions are used to compute the VaR: the historical distribution (used as a benchmark), the extreme value distribution and the normal distribution. A fully aggregated position is considered to compute the VaR with the historical distribution. In this case the history of returns on the position is repeated whenever the position weights change. To compute the VaR with the extreme value distribution and the normal distribution the decomposition of the position on a given set of risk factors (here the two stock indexes) is considered. In this case a risk-aggregation formula is used to relate the VaR of the position to the VaR of long or short positions in stock indexes, the correlation between the stock indexes and the position weights. The percentage difference between the VaR based on the historical distribution and the VaR based on the extreme value distribution or the normal distribution is reported in parentheses next to the VaR numbers. The VaR is computed with a holding period of 10 days ($f = 10$) and a value for the probability of a 10-day return not exceeding the VaR equal to 99% ($p = 0.99$) as defined by regulators (see Basle Committee, 1996a). For the extreme value method extreme 10-day returns are selected over a semester ($n = 12$ and $T = 120$). The value of the probability of an extreme return not exceeding the VaR is equal to 88.64% ($= 0.99^{12}$). All distributions are estimated over the period January 1976–December 1993.

method. In most cases the extreme value method gives similar results to the historical method and always performs better than the variance/covariance method.

5. Conclusion

In this article, I propose a new approach to computing the VaR of a market position. This approach considers extreme values and is implemented using a parametric method based on extreme value theory. This theory allows one to take into account explicitly the rare events contained in the distribution tails. As shown by the theoretical results, the general form of the asymptotic distribution of extreme returns is consistent with many statistical models for the process of returns (the normal distribution, the mixture of normal distributions, the Student distribution, the family of stable Paretian distributions, the class of ARCH processes...). The extreme value method presents three main advantages over classical methods. First, as the extreme value method is parametric, out-of-sample VaR computations are possible for high probability values. With the historical method the VaR cannot be computed for high probability values because of the limited number of observations. Second, as the extreme value method does not assume a particular model for returns but lets the data speak for themselves to fit the distribution tails, the model risk is considerably reduced. With the normal distribution or any given distribution of returns, the distribution tails may be badly fitted. Third, as the extreme value method focuses on extreme events, the event risk is explicitly taken into account. With conditional distributions considering all returns such as the GARCH or EWMA processes, large unexpected market shocks are ignored.

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